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STRAIN ENERGY METHODS OF  
STRESS ANALYSIS

## AEROPLANE STRUCTURES.

By A. J. SUTTON PIPPARD, M.B.E.,  
D.Sc., M.INST.C.E., M.I.MECH.E.,  
and Captain J. LAURENCE PRITCHARD.  
With an Introduction by L. BAIRSTOW,  
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# STRAIN ENERGY METHODS OF STRESS ANALYSIS

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*WITH DIAGRAMS*

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TO  
ROBERT MUIR FERRIER  
FIRST PROFESSOR OF CIVIL ENGINEERING  
IN  
THE UNIVERSITY OF BRISTOL  
THIS BOOK IS DEDICATED BY A FORMER STUDENT  
AS A TOKEN OF REGARD AND AFFECTION



## PREFACE

THIS book has been prepared primarily for students of engineering, especially those reading for Honours, but it is hoped that it will prove acceptable also to engineers actually engaged in structural work.

The classic work of Castigliano published in 1879 under the title *Théorie de l'équilibre des systèmes élastiques*, of which an English translation is now available,<sup>1</sup> is, on account of its detailed character, rather difficult for the student to read, especially if the subject is a new one to him, and in consequence the valuable methods of analysis described in it are perhaps not so widely known as they deserve to be.

The author of the present book hopes that it may fill a gap between the outline treatment of the subject given in many general works dealing with the theory of structures and the original treatise to which reference has been made.

In the first part of the book the various theorems are proved in a way which should appeal to the reader who finds more direct mathematical treatments difficult to follow. The second part consists of a number of examples worked out in detail, and chosen to illustrate the different applications of the methods. With one exception, these examples have been taken from papers published by the author during the last few years.

It has been assumed that the reader has a sound knowledge of methods of stress analysis as applied to just-stiff frames, and so, in general, these have not been described. It has, however, been considered desirable to include an account of the method of tension coefficients due to Mr. R. V. Southwell, F.R.S., as this method is of great service, and, so far as the author is aware, has not yet appeared in any text-book.

The Controller of His Majesty's Stationery Office has given

<sup>1</sup> *Elastic Stresses in Structures*. Translation by E. S. Andrews. Published by Scott, Greenwood and Co.

permission for the inclusion of material from various official publications, and similar permission has been given by the Council of The Institution of Mechanical Engineers for the use of papers from their *Proceedings*. The author desires to express his thanks to these authorities.

He also wishes to thank Miss H. P. Hudson, O.B.E., Sc.D., for her great help in reading the proofs.

A. J. SUTTON PIPPARD.

UNIVERSITY COLLEGE,  
CARDIFF.  
*March, 1928.*

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# STRAIN ENERGY METHODS OF STRESS ANALYSIS

## PART I CHAPTER I

### GENERAL PRINCIPLES

**Frameworks.**—A skeleton framework is an arrangement of bars pinned together to form a structure that can resist geometrical distortion under the action of any applied load system. This resistance to distortion is obtained solely by the arrangement of the bars, and since the joints are incapable of transmitting a bending moment, all the members of the frame are in a state either of pure tension or pure compression when the load system is applied only at the joints. Such internal forces or stresses are termed primary stresses.

It is unusual in an actual structure to rely upon pinned joints, and the general practice is to fasten the members of a structure together by riveted joints. These introduce stresses additional to the primary, which are known as secondary stresses. It must be clearly understood that the terms primary and secondary are not intended to give an idea of the relative magnitude or importance of the stresses, but are simply distinguishing names.

Frameworks are divided into plane frames and space frames. In a plane frame all the bars lie in one plane, and the frame is stiff against distortion for any load system applied in that plane, but not for loads out of it. A space frame, on the other hand, is a three-dimensional structure and is stiff for loads applied in any direction whatsoever.

The simplest plane frame is a triangle formed of three bars pinned at their ends. If it is desired to brace another point to this elementary frame, two more bars are needed, and every

## 2 STRAIN ENERGY METHODS OF STRESS ANALYSIS

additional point thereafter must be connected to the existing frame by two more members.

Hence, if  $j$  is the number of joints to be connected into a plane frame and  $n$  is the minimum number of bars required to effect this bracing, we have

$$n=2j-3 \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

This, while giving the minimum number of bars required, is not a sufficient criterion as to the completeness of the frame. It is clearly possible to arrange the correct number of bars in such a way that one part of the frame is overbraced, while another part is incomplete. To guard against this it is essential that the frame shall be properly triangulated, *i.e.* every joint must be attached to two braced points by two members.

The relation just given also assumes that any member in the frame is able to resist either tension or compression as required. If it happens that a member, which is required, for example, to take a compression, is actually incapable of doing so, as would be the case if it were a wire or cable, it is for practical purposes non-existent and must be ignored in determining the sufficiency of bracing.

It will be appreciated, therefore, that the relation given must be used with judgment, and can only be taken as the criterion for the least number of bars essential to the frame. If in any arrangement there are less than this relation shows to be necessary, it is certain that the frame is incomplete; if the relation is satisfied, it is still necessary to verify that proper triangulation has been effected and that all members are capable of resisting loads in the sense required by the load system.

In the case of a space frame, the unit is the tetrahedron, *i.e.* it consists of six members which connect four joints. In order to connect another joint to this elementary frame it is necessary to use three bars, and any additional joints will also require three bars each. Thus we obtain the relationship

$$n=3j-6 \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

as the criterion for the minimum number of bars required in a space frame. This relation, like that for a plane frame, must be used with attention to arrangement and suitability of the members.

**Essential and Redundant Reactions.**—A frame constructed according to the rules of the last paragraph will resist without geometrical distortion the action of any load system in equilibrium which may be applied to it. The usual function of a frame is to transmit loads to certain points, *e.g.* a roof truss must transmit

any loads due to the weight of the truss, the roof covering, and such varying loads as wind pressure may cause, to the walls or stanchions upon which it rests. In all practical cases, therefore, we may consider the external load system to be composed of two parts—the loads which the structure is designed to transmit to prepared supports, and the reacting forces called into play at these supports. It is necessary now to consider the essentials of these reacting forces. If a plane frame, which satisfies the conditions already obtained, is to be supported so that it can transmit any load system to supporting points, the reactions at these points must provide such forces that they and the applied loads are in a state of static equilibrium. The conditions of equilibrium which must be fulfilled are threefold, as follows:—

(1) The sum of the components of the applied loads parallel to any axis  $Ox$  in the plane of the frame must be equal and opposite to the sum of the components of the reactions parallel to the same axis.

(2) The sum of the components of the applied loads parallel to another axis  $Oy$  at right angles to  $Ox$ , also in the plane of the frame, must be equal and opposite to the sum of the components of the reactions parallel to the same axis.

(3) The moment of all the applied forces about any point in the plane must be equal and opposite to the moment of the reactions about the same point.

These conditions provide three equations of equilibrium, and so the reactions may consist of three components.

Conditions (1) and (3) can be satisfied by providing forces parallel to the axis  $Ox$  at two points, but in order to satisfy (2) it is necessary to provide at one of these points a force parallel to the axis  $Oy$ . Thus, two reacting forces parallel to one axis, and one reacting force parallel to the perpendicular axis are a complete and sufficient system of supporting forces.

A pin joint can transmit a force in any direction, so that, if we make one of our supports a pin, this is able to provide components parallel to the two axes, and if another point of the frame is held on a roller bearing or other frictionless support, so that it can react parallel to one axis only, the frame is properly supported against any applied load system; thus the correct reactions are automatically provided, and these reactions can be evaluated by the methods of statics. If, however, two points of the frame are pinned to supports, each pin can exert reactive forces parallel to both axes, and there are four instead of three terms in the three static equations, which are then indeterminate. As an illustration, consider the common case of the king-post truss,

which is a just-stiff frame. If this truss is pinned to one wall and rests on rollers at the other wall, all the horizontal components of the applied loads are balanced at the pin, and the values of the vertical reactions are readily calculable from the other two equations. If, however, both points of the truss are pinned to wall supports the horizontal forces are balanced partly at one pin and partly at the other, and the equations are insufficient to determine the proportions. Such a frame is said to have a redundant reaction, and to find its value strictly requires the application of strain energy methods. It is customary, however, in such cases as this to assume that the horizontal reactive forces at the two pins are equal. As will be shown in the next paragraph, one member of the king-post truss can be dispensed with in the case where two pinned supports are provided.

When the frame is a space frame the conditions of static equilibrium to be satisfied are as follows :—

(1) The components parallel to any axis  $Ox$  in space must balance.

(2) The components parallel to an axis  $Oy$ , perpendicular to  $Ox$ , must balance.

(3) The components parallel to an axis  $Oz$ , perpendicular to both  $Ox$  and  $Oy$ , must balance.

(4) The moments about the axis  $Ox$  must balance.

(5) The moments about the axis  $Oy$  must balance.

(6) The moments about the axis  $Oz$  must balance.

There are thus six equations to be satisfied by the reactive forces, and if more than six component forces are provided at the supporting points, any above this number are redundant, *e.g.* if three points are pinned to supports, each is able to exert three component reactive forces and the frame has three redundant reactions. In a space frame it is only necessary for one point of support to be pinned, for another to provide constraint in a plane, and for a third to provide constraint against movement out of the plane. If more than these are provided, the frame can dispense with some of its members, as will now be shown.

**Frames supported at a Number of Pinned Points.**—Suppose that a plane frame having  $(J+j)$  joints is pinned to a rigid body, *e.g.* to a wall, at  $j$  of these joints. The wall serves to brace these  $j$  joints together, and so may be considered as being equivalent to  $2j-3$  members.

To brace all the joints together it would be necessary to provide  $2(J+j)-3$  members, so that to connect the  $J$  free joints to the  $j$  pins it is necessary to provide  $\{2(J+j)-3\}-\{2j-3\}$  or  $2J$  members.

Hence we get the rule that for a plane frame  $2J$  members are needed to brace  $J$  joints to any number of fixed points.

Similarly, if the structure is a space frame of  $(J+j)$  joints,  $j$  of which are pinned supports, the "wall" is equivalent to  $3j-6$  members, while the total number required is  $3(J+j)-6$ . Hence, the number of bars required to brace the  $J$  free joints to the  $j$  pins is  $3J$ , and so for a space frame the rule is obtained that  $3J$  members are needed to brace  $J$  points to any number of pinned supports.

Fig. 1 illustrates the case of a plane frame.

At (a) we have seven joints which are to be connected to the two points A and B on the wall. From the rule just obtained it is seen that this needs fourteen members which may be provided as shown in the figure.

At first sight this frame does not appear to be stiff, but examination will show that it is effectively braced, thus: AD

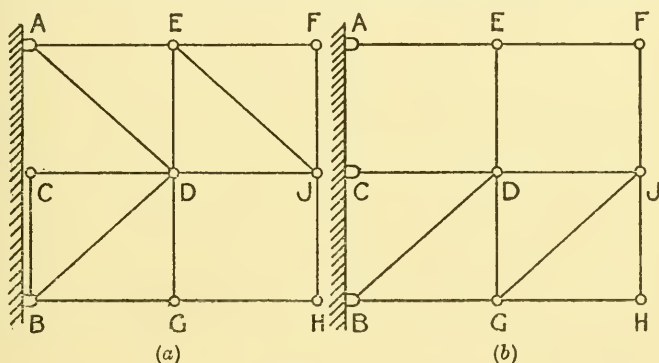


FIG. 1.

and BD fix the point D, and DC and CB fix C. Joints E and G are clearly braced, and so are J and F. The remaining joint H is fixed by JH and GH.

At (b) is shown a case where six joints are to be fixed to three points on the wall A, B, and C. This needs twelve members, provided as shown. D and G are clearly fixed to C and B, while E is fixed by DE and AE, J by DJ and GJ, and H by GH and HJ. Finally, F is fixed by JF and EF.

Now consider the case of a space frame shown in Fig. 2.

At (a) nine joints are shown which are to be connected to the three points A, B, and C on the wall. This requires twenty-seven members. These are provided by eight longitudinal members, the transverse members AD and CD, eight other transverse members, eight diagonal members—one in each longitudinal panel—and one transverse diagonal EF.

If now the four points A, B, C, and D are fixed as shown at (b), and the remaining eight points are to be connected to them, we need twenty-four bars, *i.e.* three less than in case (a). The three members that can be dispensed with are AD, CD, and EF, and the frame is completely braced by the eight longi-

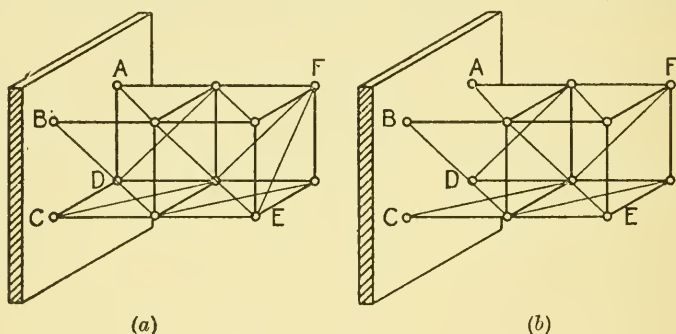


FIG. 2.

tudinals, eight transverse members and eight panel members, as shown in the figure.

**Just-Stiff and Redundant Frames.**—Frames which just satisfy the conditions given in the preceding paragraphs are said to be “just-stiff,” and the stress analysis of such frames can be carried out by the methods of statics. If the number of bars is less than that given by these conditions, the frame is said to be incomplete. This is something of a misnomer, since, if the joints are pinned, such a collection of bars cannot be called a frame, as it would collapse except for particular types of loading. It can only be considered to be a structure if it is provided with stiff joints which take the place of the missing members. An arrangement of bars which is one short of the number required to make it just-stiff is a mechanism.

If the number of bars provided is greater than that required by the conditions of sufficiency for stiffness, the frame is said to be redundant. In this case the methods of statics by themselves are not sufficient to analyse the stress distribution, and in consequence such frames are often called “statically indeterminate” frames. It is in such cases that the methods of strain energy are useful in obtaining the stress distribution.

**Pseudo-Redundant Frames.**—In some cases it may appear that a frame is redundant when in reality it is just-stiff. As an example consider the plane frame shown in Fig. 3.

This frame has eight joints to be braced to two points on a wall, and to do this sixteen members are required. Actually

it contains twenty members, and would appear to be redundant to the extent of four members. If all the diagonal members were capable of taking either tension or compression, this would be correct, but suppose that these members consist of tension bars of small flexural rigidity. They are unable to resist compression, and when called upon to do so they will slacken and the whole shear across a section will be thrown upon the tension diagonal. It is clear, then, that for a down load as shown in the diagram, four of the diagonals will be inoperative, while for an upward load the other four will slacken. So for any condition of loading the structure is just-stiff. This type of frame is said to be pseudo-redundant, and care must be taken to detect such structures before beginning a stress analysis.

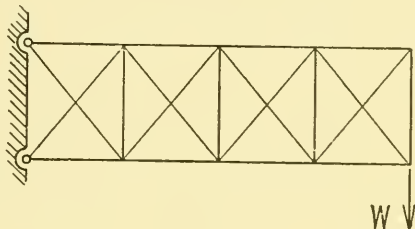


FIG. 3.

**Initially Tensioned Ties and Slender Struts.**—In the preceding paragraph reference was made to the fact that tension rods might not be able to take compressive forces. In general this is true, but in certain cases they are able to act as struts.

In many instances where structures contain tension wires or cables, the structure is assembled and these cables then tightened up so that they are subjected to tension before the external load is applied. Suppose such a member has an initial tension of magnitude  $T$ , and that under the action of the external load system it is required to resist a compression  $C$ . As long as  $T$  is greater than  $C$ , the member has a tensile load in it of  $(T-C)$ , and will function. If the load  $C$  is equal to  $T$ , the cable is just tight, but if  $C$  is greater than  $T$  the member becomes inoperative.

If instead of an initially tensioned wire the member is a long slender strut, it remains operative until the load in it reaches the critical value. After this, no increase in the external loading affects the load in it, since it has reached the limit, and even although it has bowed slightly it still exerts a thrust equal to its critical load.

**Self-straining.**—Since a frame with one member absent is a mechanism, an error in the length of the absent member, when it is put into its position in the frame, is unimportant and will occasion no trouble in assembly. The only effect upon the frame will be a slight alteration in its configuration. If, however, a just-stiff frame is to be fitted with a redundant member, it is

necessary to make this member of the exact length, since the two joints to which it is to be attached are already fixed in position relatively to each other. If by error or design the member is not of the exact length, force will have to be exerted to get it into position—the two points to be joined will have to be brought closer together or forced apart. This means that before any external load is applied to the frame, its members are in a state of stress. This action is known as self-straining, and any forces in the members due to initial stresses must be added to those caused by the action of the external load system.

**Stress Analysis of Just-Stiff Frames by Tension Coefficients.**—The method of sections and the stress diagram are both simple means of determining the distribution of internal forces in a plane

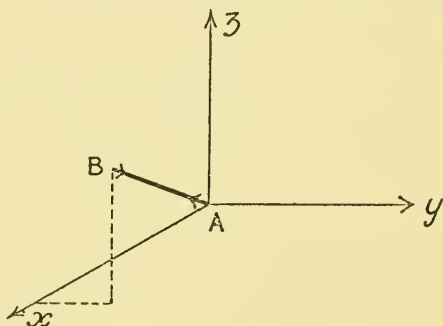


FIG. 4.

frame under load. These methods are fully described in all text-books on the theory of structures, and it is unnecessary to repeat the descriptions here. Their application to a space frame is, however, not an easy operation, especially if the frame is at all complicated, and when such structures have to be dealt with, the method known as that of tension coefficients, due to Mr. R. V. Southwell,<sup>1</sup> is undoubtedly of great value. Since this treatment is not yet generally known, it is described here, and although it is of most value in the treatment of space frames its use is in no way restricted to these, and it can be used with advantage in many cases of plane frames.

Let A and B (Fig. 4) be two joints in a space frame connected by the bar AB.

Through A take three mutually perpendicular axes  $Ax$ ,  $Ay$ , and  $Az$ .

<sup>1</sup> "Primary Stress Determination in Space Frames." R. V. Southwell. *Engineering*, Feb. 6, 1920.

Let the *tension* in AB be  $T_{AB}$ .

If the length of AB be  $l_{AB}$  it will be found convenient to write

$$T_{AB} = l_{AB} \cdot t_{AB},$$

where  $t_{AB}$  is defined as the tension coefficient of the member AB.

The component of the tension in AB along the axis  $Ax$  is a pull on the joint A of magnitude  $T_{AB} \cos \widehat{BAx} = t_{AB} \cdot l_{AB} \cos \widehat{BAx}$ .

Now 
$$\cos \widehat{BAx} = \frac{x_B - x_A}{l_{AB}},$$

where  $x_A$  and  $x_B$  are the  $x$  co-ordinates of the points A and B.

Hence the pull on the joint A along the  $x$  axis is  $t_{AB}(x_B - x_A)$  and the pull on the joint B along the  $x$  axis is  $t_{AB}(x_A - x_B)$ .

At any point A let the components of the external forces be  $X_A$ ,  $Y_A$ , and  $Z_A$  acting along the  $x$ ,  $y$ , and  $z$  axes respectively.

Also let the joints B, C . . . Q be connected to the joint A by members of the frame.

Then the equations of equilibrium of the joint A can be written in the form—

$$\begin{aligned} t_{AB}(x_B - x_A) + t_{AC}(x_C - x_A) + \dots + t_{AQ}(x_Q - x_A) + X_A &= 0 \\ t_{AB}(y_B - y_A) + t_{AC}(y_C - y_A) + \dots + t_{AQ}(y_Q - y_A) + Y_A &= 0 \\ t_{AB}(z_B - z_A) + t_{AC}(z_C - z_A) + \dots + t_{AQ}(z_Q - z_A) + Z_A &= 0 \end{aligned}$$

Three equations of this sort can be written down for each joint of the frame, and it will be observed that the terms  $(x_B - x_A)$ ,  $(y_B - y_A)$ , etc., are the projected lengths of the members, which are usually dimensioned in working drawings, so that all the data necessary are given on such drawings in a form ready for use.

The unknowns in these equations are the tension coefficients, and as each of these occurs in the equations for two joints no difficulty is experienced in finding the values, as will be seen from the example following.

Having found the values of the tension coefficients, the loads in the bars are obtained by multiplying them by the lengths of the members, *e.g.*

$$T_{AB} = t_{AB} \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2 + (z_B - z_A)^2}.$$

Where the frame under consideration is plane the work is very much simplified, since there are no terms for the  $z$  axis.

As an example of this method we will analyse the space frame shown in Fig. 5, which consists of a cantilever structure formed of three longitudinal members ABC, GHJ, and DEF braced together and arranged so that at any section they lie at the corners of an equilateral triangle.

As there are six points to be braced to the wall, eighteen

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members are necessary. These are provided by the six longitudinal sections, six struts, and six diagonal panel members.

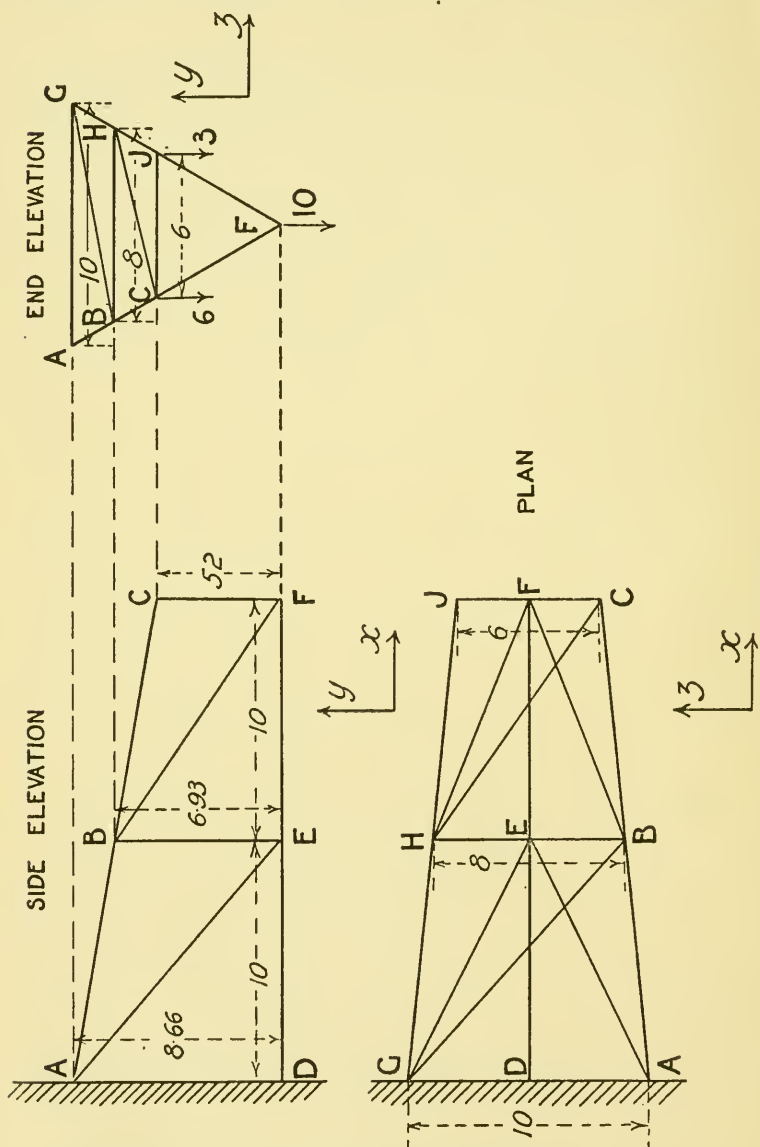


Fig. 5.

The loading is as shown on the diagram.

Taking the co-ordinate axes positive in the directions indicated,

and starting with point F, we can write down the equations of equilibrium as follows:—

The members meeting at this joint are FE, FC, FJ, FH, FB, and if we consider first the equation for the  $x$  axis, it is clear that FC has no component.

The projection of FE on this axis is  $-10$ , that of FB is  $-10$ , and that of FH is  $-10$ , so that the equation becomes

$$-10t_{FE}-10t_{FB}-10t_{FH}=0$$

Considering now the  $y$  axis, the members concerned are FC, FJ, FB, and FH. The projections on the  $y$  axis of FC and FJ are both  $5.2$ , and of FB and FH,  $6.93$ . The load is  $10$  acting in a negative direction, so that the equation becomes

$$5.2t_{FC}+5.2t_{FJ}+6.93t_{FB}+6.93t_{FH}=10.$$

Similarly, for the  $z$  axis we get

$$3t_{FJ}-3t_{FC}+4t_{FH}-4t_{FB}=0.$$

Following the same procedure at every joint, we get the necessary equations, which are best tabulated as shown in Table I.

TABLE I.

Joint.		Equations.
F	$x$	$-10t_{FE}-10t_{FB}-10t_{FH}=0.$
	$y$	$5.2t_{FC}+5.2t_{FJ}+6.93t_{FB}+6.93t_{FH}-10=0.$
	$z$	$3t_{FJ}-3t_{FC}+4t_{FH}-4t_{FB}=0.$
C	$x$	$-10t_{CB}-10t_{CH}=0.$
	$y$	$-5.2t_{CF}+1.73t_{CB}+1.73t_{CH}-6=0.$
	$z$	$6t_{CJ}+7t_{CH}-t_{CB}+3t_{CF}=0.$
J	$x$	$-10t_{JH}=0.$
	$y$	$1.73t_{JH}-5.2t_{JF}-3=0.$
	$z$	$-6t_{JC}+t_{JH}-3t_{JF}=0.$
E	$x$	$10t_{EF}-10t_{ED}-10t_{EA}-10t_{EG}=0.$
	$y$	$6.93t_{EB}+8.66t_{EA}+6.93t_{EH}+8.66t_{EG}=0.$
	$z$	$4t_{EH}-4t_{EB}+5t_{EG}-5t_{EA}=0.$
B	$x$	$10t_{BC}-10t_{BA}-10t_{BG}+10t_{BF}=0.$
	$y$	$-6.93t_{BE}+1.73t_{BA}+1.73t_{BG}-1.73t_{BC}-6.93t_{BF}=0.$
	$z$	$8t_{BH}+9t_{BG}+t_{BC}-t_{BA}+4t_{BE}+4t_{BF}=0.$
H	$x$	$10t_{HJ}-10t_{HG}+10t_{HC}+10t_{HF}=0.$
	$y$	$1.73t_{HG}-1.73t_{HJ}-1.73t_{HC}-6.93t_{HE}-6.93t_{HF}=0.$
	$z$	$-8t_{HB}-7t_{HC}-t_{HJ}+t_{HG}-4t_{HF}-4t_{HE}=0.$

These equations must now be solved. In the present instance, joint J is taken first, and the equation  $Jx$  for forces along the  $x$

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axis gives  $t_{JH}=0$ . Substituting this value in  $Jy$ , we find  $t_{JF}$  to be  $-0.576$ , and then  $t_{JC}=0.288$  follows at once from  $Jz$ .

If joint C be now considered, it will be seen from  $Cx$  that  $t_{CB}+t_{CH}=0$ , which can be substituted in  $Cy$  to obtain  $t_{CF}=-1.15$ . If we use this and the known value of  $t_{CJ}$  in  $Cz$ , an equation in  $t_{CH}$  and  $t_{CB}$  is obtained, which with  $Cx$  enables these two coefficients to be evaluated.

This process is continued until all the coefficients have been determined, and these multiplied by the respective lengths of the members give the loads in the members as set out in Table II.

TABLE II.

Member.	Tension-coefficient $t$	Length $L$	Load $Lt$
FC	-1.15	6.0	-6.9
FJ	-0.576	6.0	-3.46
FB	1.59	12.8	20.3
FE	-2.75	10.0	-27.5
FH	1.16	12.8	14.85
CB	-0.215	10.2	-2.19
CJ	0.288	6.0	1.73
CH	0.215	12.3	2.64
JH	0	10.2	0
EB	-1.194	8.0	-9.55
ED	-4.4	10.0	-44.0
EA	0.961	14.14	13.60
EG	0.684	14.14	9.7
EH	-0.86	8.0	-7.08
BA	1.28	10.2	13.06
BG	0.133	13.56	1.80
BH	-0.166	8.0	-1.328
HG	1.375	10.2	14.02

**The Principle of Superposition.**—Suppose a bar to carry a tensile load of  $(P+Q)$  lbs.

Let  $E$  be the value of Young's modulus for the material,

$A$  its cross-sectional area,

$L$  its length.

Then if the material obeys Hooke's law, the extension of the

$$\text{bar is } e = \frac{(P+Q)L}{AE} = \frac{PL}{AE} + \frac{QL}{AE}.$$

That is, the total extension of the bar under the load  $(P+Q)$  is the sum of the extensions under the two loads  $P$  and  $Q$  considered separately. This is known as the principle of superposition.

It is illustrated graphically in Fig. 6 (a), where a load-extension diagram for a material obeying Hooke's law is given.

Under a load  $P$  the extension is  $OA$ , and under a load  $Q$  it is  $OB$ . When  $(P+Q)$  is applied the extension is  $OC$ .

Now the triangles  $OBE$  and  $FGH$  are identical, since

$$BE = GH = Q, \text{ and so } OB = FG.$$

Now  $OC = OA + AC = OA + FG = OA + OB$

*i.e.* the extension of the bar due to the load  $(P+Q)$  is equal to the extension due to  $P$  + the extension due to  $Q$ .

Now consider the case shown at (b) where the load extension curve is no longer a straight line, *i.e.* the material does not follow Hooke's law. The separate extensions of the bar under the loads  $P$  and  $Q$  respectively are  $OA$  and  $OB$ . Under the combined load  $(P+Q)$  the extension is  $OC$ , and it is clear from the diagram that  $OB$  is now not equal to  $AC$ , and we can no longer find the

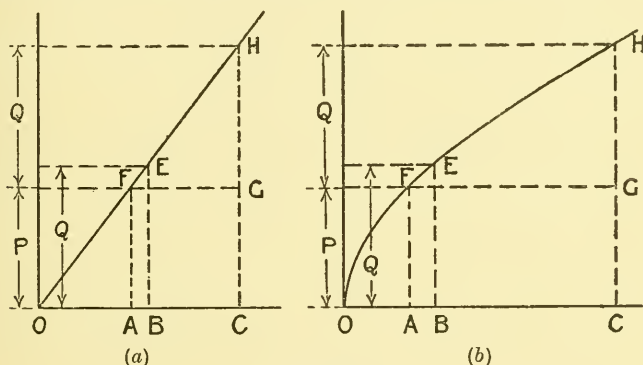


FIG. 6.

effect of  $P$  and  $Q$  acting together by combining their separate effects.

When a framework consists of bars which obey Hooke's law, the frame considered as a whole also follows the law, and the principle of superposition can be applied, *e.g.* if it is desired to find the deflection of a point  $A$  under the action of any number of loads  $W_1, W_2, W_3, \dots, W_N$ , it can be done by calculating the deflection of  $A$  under each load separately, and then adding these separate deflections.

Further, the strains of individual members of the frame can be found by superposing the effects of the single loads, and so the internal stress in a bar under the complete load system is the algebraic sum of the stresses due to the separate loads.

This result is of great importance, and it must be emphasized that it is only applicable when the whole structure obeys Hooke's law. *The whole of the theorems relating to strain energy, which*

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*will be dealt with later, depend for their validity upon the applicability of the principle of superposition, and if this cannot be applied it is incorrect to use the methods of strain energy.*

**Frameworks and Elastic Solids.**—In a just-stiff frame the stress distribution depends only upon the geometry of the frame, but, as will be shown later, the deflections are dependent upon the elastic properties of the members of the structure as well as upon its geometry. In a redundant frame both the stress distribution and the distortion under load depend upon the elastic properties of the frame. As a frame is made increasingly redundant by the addition of more and more members it approaches the condition of a solid, and an elastic solid may be considered as the limiting case of a redundant structure. It follows that any properties of a frame which are dependent upon its elasticity are also possessed by an elastic solid. In the work that follows, therefore, it is to be understood that, unless stated to the contrary, theorems connected with elastic properties which are applicable to a frame are also applicable to an elastic solid.

**Saint Venant's Principle.**—An important principle enunciated by Saint Venant states that the strains which are produced in an elastic solid, by the application to a small portion of its surface of a system of forces statically equivalent to zero force and zero couple, are of negligible dimensions at distances which are large compared with the linear dimensions of that portion.

This principle may be combined with the principle of superposition, and can then be restated as follows: if any systems of loads which are statically equivalent, *i.e.* which produce the same resultant forces and couples, are applied separately to the same small part of an elastic body, then the strains, and as a consequence the stresses, produced at a distance from the part to which the loads are applied, are the same for all of these statically equivalent systems.

This principle was stated by Saint Venant without proof, but all experimental evidence since has gone to prove that it is valid, and moreover that the distance from the part of application of the loads at which it operates is, for practical purposes, a very small one.

A demonstration of the truth of this principle has been given by Mr. R. V. Southwell, based on considerations of strain energy.<sup>1</sup>

Since an elastic solid is in reality the limiting case of a redundant frame, it appears reasonable to suppose that the principle of Saint Venant applies to the case of such frames when the

<sup>1</sup> "On Castigliano's Theorem of Least Work and the Principle of Saint Venant," *Phil. Mag.*, January, 1923.



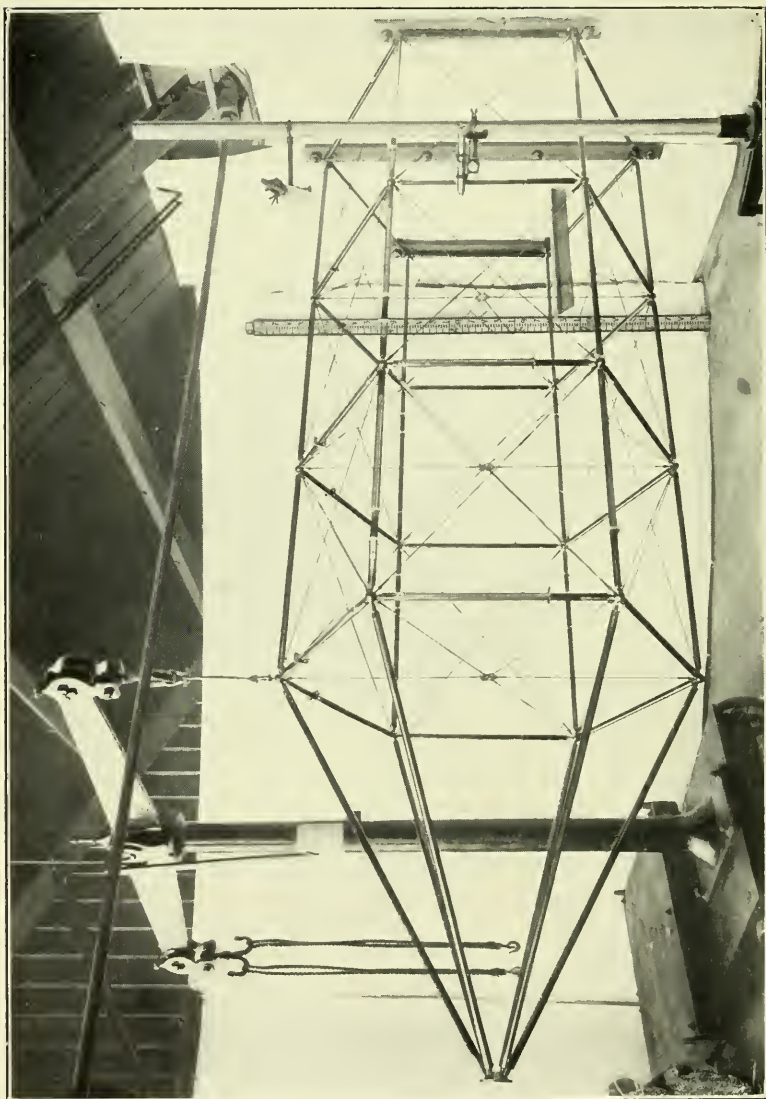


FIG. 7.

small portion of the solid is replaced by a set of adjacent joints. The extent of its validity must, of course, depend upon the degree of redundancy; it will be appreciated that in a just-stiff frame, where the stress distribution depends only upon the geometry of the structure and the method of application of the loads, it cannot apply, and that in a frame with a finite number of redundancies much must depend upon their number and disposition.

In the course of a research upon the properties of certain redundant frames, work was carried out to test the applicability of this principle to frameworks. The structure used was a braced hexagonal tube shown in Fig. 7, and the full account of the experiments will be found in the original papers.<sup>1</sup>

The load was applied to the end bulkhead, *i.e.* the one furthest from the wall support, and a number of different but statically equivalent loadings were used. In addition, the conditions of redundancy were modified, and the following general conclusions were reached :—

1. The principle of Saint Venant can be applied to the case of a redundant framework.

2. The operation of the principle is very slow if planes parallel to the loading plane are unbraced, even when the framework exhibits a high degree of redundancy in other planes.

3. If the plane of loading is braced, the tendency towards the equalization of strains is pronounced, and the distance required for the variation of strains to become negligible is dependent to a marked extent upon the efficiency of the bracing in this and parallel planes.

<sup>1</sup> "On an Experimental Verification of Castigliano's Principle of Least Work and of a Theorem relating to the Torsion of a Tubular Framework." A. J. S. Pippard and J. F. Baker. *Phil Mag.*, July, 1925.

"On an Experimental Investigation of the Applicability of Saint Venant's Principle to the Case of Frameworks having Redundant Bracing Members." A. J. S. Pippard and G. H. W. Clifford. *Phil. Mag.*, January, 1926.

## CHAPTER II

### GENERAL THEOREMS RELATING TO STRAIN ENERGY

**Strain Energy.**—When an elastic body is strained within the limit of proportionality, it stores energy which is recoverable on the removal of the straining actions. This energy is known as strain energy.

**Strain Energy in Direct Tension or Compression.**—Fig. 8 shows the load extension curve for a tensile specimen made of material which obeys Hooke's law.

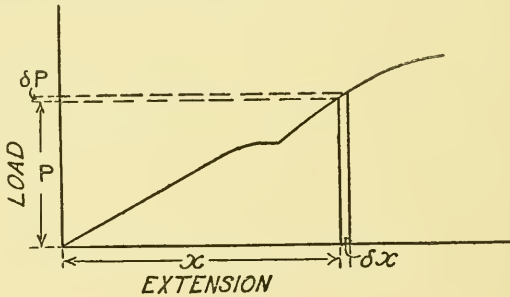


FIG. 8.

At any point on this curve let the extension be  $x$ , and the corresponding load  $P$ .

If the load is increased by a small increment  $\delta P$ , the extension is increased by a small amount  $\delta x$ , so that the final values of the load and extension are  $(P + \delta P)$  and  $(x + \delta x)$  respectively.

The work done by the load during the small increase of extension is

$$(\text{average value of load}) \times (\text{distance moved through})$$

that is 
$$\left(P + \frac{\delta P}{2}\right) \times (\delta x).$$

Neglecting second order quantities, we get  $P\delta x$ . Hence the total work done on the specimen as the load increases from  $P_1$  to  $P_2$  is

$$\int_{P_1}^{P_2} P dx,$$

which is the area of the load extension curve between the two values of the load  $P_1$  and  $P_2$ .

If we confine our attention now to the part of the curve which is linear we can write

$$x = \frac{PL}{AE},$$

for values of  $P$  within the limits of proportionality, where  $L$  is the length of the specimen and  $A$  is its cross-sectional area.

Then 
$$\delta x = \frac{L}{AE} \delta P,$$

and the total work done on the specimen, as the load is increased from zero to  $P_0$ , is given by

$$u = \int_0^{P_0} \frac{PL}{AE} dP = \frac{1}{2} \frac{P_0^2 L}{AE}.$$

This result is obtained on the assumption that the load is applied in such a way that there is no kinetic energy set up, and so the total work done by the load is stored as potential or strain energy in the bar.

Hence the strain energy of a bar subjected to a pure tension or compression  $P_0$  is

$$u = \frac{1}{2} \frac{P_0^2 L}{AE} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

If the load had been applied in any other manner, kinetic energy would be supplied to the bar, and a temporary extension would occur greater than the value  $\frac{PL}{AE}$ . The extra energy would be used up in causing the bar to vibrate about its static extension position, but after these vibrations had been completely damped out, the energy remaining in the bar would be the same as if the load had been so applied that no kinetic energy were set up.

It is evident, therefore, that the method of applying the load is immaterial; the strain energy of the bar depends only on the final value of the load and not on the previous history of the specimen.

**Internal and External Work.**—Let  $AB$  and  $BC$  be two members inclined at angles  $\theta$  and  $\phi$  to the  $x$  axis and pinned together at  $B$  (Fig. 9).

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Let a vertical load  $Y$  and a horizontal load  $X$  act on the frame at  $B$ . Under these loads let  $B$  move a distance  $a$  parallel to the  $x$  axis and a distance  $b$  parallel to the  $y$  axis.

Before strain let the co-ordinates of  $A$ ,  $B$ , and  $C$  be

$$(x_A, y_A), (x_B, y_B) \quad \text{and} \quad (x_C, y_C).$$

After strain they will be

$$(x_A, y_A), (x_B + a, y_B + b), \quad \text{and} \quad (x_C, y_C).$$

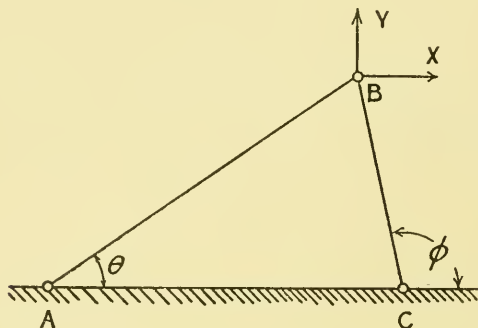


FIG. 9.

Then the strain of  $AB$  will be

$$e_{AB} = \frac{a(x_B - x_A) + b(y_B - y_A)}{L_{AB}^2}$$

and the strain of  $BC$  will be

$$e_{BC} = \frac{a(x_B - x_C) + b(y_B - y_C)}{L_{BC}^2}.$$

Hence the load in  $AB$  will be

$$T_{AB} = \frac{EA_{AB}}{L_{AB}} \left\{ \frac{a(x_B - x_A) + b(y_B - y_A)}{L_{AB}} \right\}$$

and that in  $BC$ ,

$$T_{BC} = \frac{EA_{BC}}{L_{BC}} \left\{ \frac{a(x_B - x_C) + b(y_B - y_C)}{L_{BC}} \right\},$$

or

$$\frac{T_{AB} L_{AB}}{EA_{AB}} = a \cos \theta + b \sin \theta$$

and

$$\frac{T_{BC} L_{BC}}{EA_{BC}} = a \cos \phi + b \sin \phi.$$

Multiplying these by  $\frac{1}{2}T_{AB}$  and  $\frac{1}{2}T_{BC}$  respectively and adding, we get

$$\begin{aligned} \frac{1}{2} \frac{T_{AB}^2 L_{AB}}{EA_{AB}} + \frac{1}{2} \frac{T_{BC}^2 L_{BC}}{EA_{BC}} \\ = \frac{a}{2} \{T_{AB} \cos \theta + T_{BC} \cos \phi\} + \frac{b}{2} \{T_{AB} \sin \theta + T_{BC} \sin \phi\} \end{aligned}$$

Now consider the equilibrium of the point B.

By resolution of forces along the  $x$  and  $y$  axes we get

$$T_{AB} \cos \theta + T_{BC} \cos \phi - X = 0$$

$$T_{AB} \sin \theta + T_{BC} \sin \phi - Y = 0.$$

Substituting these in the expression just obtained, we find

$$\frac{1}{2} \left\{ \frac{T_{AB}^2 L_{AB}}{EA_{AB}} + \frac{T_{BC}^2 L_{BC}}{EA_{BC}} \right\} = \frac{a}{2} X + \frac{b}{2} Y.$$

Now the left-hand side of this equation is the sum of the strain energies of the two bars AB and BC, while the right-hand side is the total work done by the external forces X and Y in moving through their respective displacements, if it is assumed that they rose to their final values at the same rate as the component displacements.

Hence

internal strain energy = external work.

It will be noticed that in arriving at this result no assumptions were made as to the manner in which the loads were applied. The strain energy is thus once more seen to depend only on the final values of the loads and their displacements, and not on the manner of their application.

The foregoing demonstration can be generalized for the case of a frame with any number of bars and external loads, and we may write

$$\sum \frac{1}{2} \frac{P_0^2 L}{AE} = \sum \frac{1}{2} W \Delta \quad \dots \quad (4)$$

where  $P_0$  is the load in any member and  $W$  is any external force which has a component displacement  $\Delta$  in its own line of action.

### Strain Energy due to Bending.

—Let the bending moment at any point in a beam be  $M$  (Fig. 10).

As in the case of direct loading it can be shown that it is

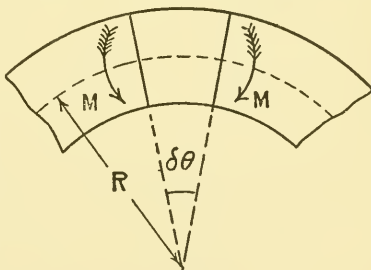


FIG. 10.

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immaterial in what manner the load is applied, so that for the sake of convenience it will be assumed to be applied gradually.

Let a small portion of the beam at the point considered subtend an angle  $\delta\theta$  at the centre of curvature.

Let the radius of curvature be  $R$ .

Then the work done by the moment during its growth from zero to its final value  $M$  is

$$\delta U = \frac{1}{2} M \delta\theta.$$

If  $\delta s$  is the length of arc subtended by  $\delta\theta$ ,

$$R \delta\theta = \delta s,$$

and 
$$\delta U = \frac{M \delta s}{2R}.$$

From the ordinary equations of bending,

$$\frac{1}{R} = \frac{M}{EI}.$$

Hence

$$\delta U = \frac{M^2 \delta s}{2EI},$$

and

$$U = \int \frac{M^2 ds}{2EI} \quad \dots \dots \dots (5)$$

which is the strain energy of a beam due to the bending.

**Strain Energy due to Shear.**—Let  $AC$  and  $BD$  be two sections of a beam under the action of a shearing force  $S$  (Fig. 11).

The distance between these sections is  $\delta x$ , and under the action of  $S$ ,  $B$  moves to  $B'$  and  $D$  to  $D'$ , relative to the section  $AC$ .

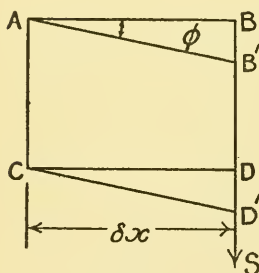


FIG. 11.

Then  $\delta U = \frac{1}{2} \cdot S \cdot BB'$

But  $BB' = \phi \delta x,$

where  $\phi$  is the shear strain, and

$$\phi = \frac{\text{shear stress}}{N},$$

where  $N$  is the modulus of rigidity of the material.

If  $A$  is the cross-sectional area of the face on which  $S$  acts,

$$\text{Shear stress} = \frac{S}{A},$$

assuming that the shear stress is uniformly distributed over the surface.

Hence 
$$\delta U = \frac{1}{2} \cdot S \cdot \frac{S}{AN} \cdot \delta x$$

and 
$$U = \int \frac{S^2 dx}{2AN} \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

In fact, the shear stress is not uniformly distributed over the cross-section, and in consequence

$$U = k \int \frac{S^2 dx}{2AN} \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

where  $k$  is a coefficient depending on the shape of the section and the loading.

**Strain Energy due to Torque.**—Suppose a torque  $T$  to be applied to a cylindrical bar as shown in Fig. 12, and let  $\delta\theta$  be the relative twist of two sections  $\delta x$  apart.

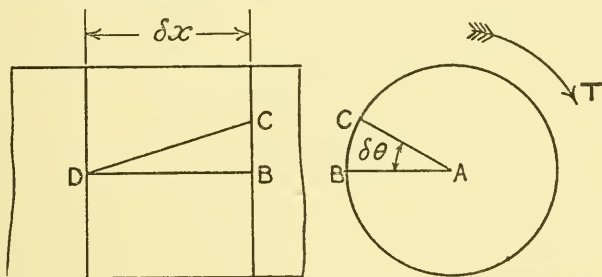


FIG. 12.

If the outer radius of the bar is  $R$ , the shear strain  $\phi$  is given by

$$\phi = \frac{BC}{DB} = \frac{R\delta\theta}{\delta x},$$

or 
$$\delta\theta = \frac{\phi\delta x}{R},$$

but 
$$\phi = \frac{q}{N} = \frac{TR}{NJ},$$

where  $q$  is the shear stress at the outer fibre and  $N$  is the modulus of rigidity of the material, while  $J$  for a circular cross-section is the polar moment of inertia.

Hence 
$$\delta\theta = \frac{T\delta x}{NJ}.$$

The work done by the torque on the small element of length  $\delta x$  is  $\frac{1}{2}T\delta\theta$ . Therefore

$$\delta U = \frac{1}{2} T \delta \theta = \frac{1}{2} \frac{T^2 \delta x}{N J}$$

or

$$U = \int \frac{T^2 dx}{2N\mathbf{J}} \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \quad (8)$$

In the case of non-cylindrical members  $J$  is not the polar moment of inertia, but has a modified value on account of the distortion of cross-sections of the bar under torque. In such cases it may sometimes be calculated,<sup>1</sup> but often it is simpler to obtain experimental values.

**The Deflection of an Elastic Body under Load.**—The fact that the strain energy is equal to the external work can be used to determine the deflection of the loaded point in the simple case when an elastic body, either a frame or a solid, carries only one load.

Suppose that the elastic body is represented by the frame

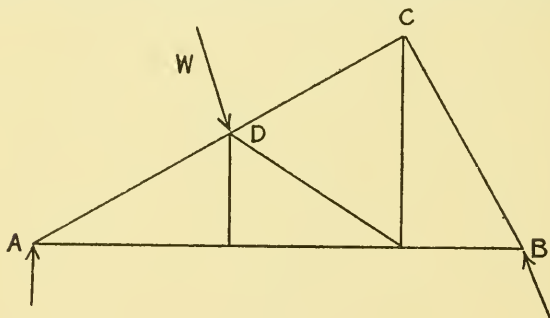


FIG. 13.

shown in Fig. 13, which is supported at A and B, and carries a single load W at the point D. *If the supports at A and B are such that the reactions do no work*, the total work done on the frame will be that due to W.

This proviso with regard to the supports is an important one in this and other problems connected with the application of strain-energy methods, and its fulfilment is ensured if the supports are fixed, in which case the points of action of the reacting forces do not move at all, or if they are roller bearings so that there is no frictional effect. In this case the reactions are normal to the bearing, and having no component movement in their line of action can do no work.

<sup>1</sup> "The Determination of Torsional Stresses in a Shaft of any Cross-section," Bairstow and Pippard. *Proc. Inst. C.E.*, vol. cexiv, 1921-22.  
*Applied Elasticity*. Timoshenko and Lessels. Chap. II. East Pittsburgh, Pa, 1925.

If the frame is just-stiff, all the internal stresses can be found by one of the standard methods of analysis; if it is redundant, the loads must be found by an application of methods to be described later.

Assuming that these internal loads have been found in terms of the weight  $W$ , we can write for the load in any member

$$P_0 = \alpha W,$$

where  $\alpha$  is a numerical coefficient.

The strain energy of the frame is

$$\frac{1}{2} \sum \frac{P_0^2 L}{AE} = \frac{1}{2} \sum \frac{\alpha^2 W^2 L}{AE}$$

and the external work done on the frame is

$$\frac{1}{2} W \Delta,$$

where  $\Delta$  is the movement of the point  $D$  in the line of action of  $W$ .

Equating the internal strain energy and the external work we find

$$\frac{1}{2} W \Delta = \frac{1}{2} W^2 \sum \frac{\alpha^2 L}{AE},$$

or

$$\Delta = W \sum \frac{\alpha^2 L}{AE} \quad . \quad . \quad . \quad . \quad . \quad (9)$$

It will be seen that  $\alpha$  is the stress in any member due to a unit load placed at the point  $D$ .

This method, while useful for bodies subjected to a single load, is of very limited application, and a more general treatment will be dealt with shortly.

**Clerk Maxwell's Reciprocal Theorem.**—Before proceeding to give a more general method for the calculation of the deflections, it is necessary to prove a theorem due in the first place to Clerk Maxwell, known as the reciprocal theorem.

Fig. 14 shows a frame and an elastic solid each carrying two loads  $W_1$  and  $W_2$  at points  $A$  and  $B$  respectively. The reacting forces  $R_1$  and  $R_2$  as before are assumed to have no component movement in their own line of action. The argument following applies equally to both the frame and the solid body. Let the

- movement of  $A$  in the direction of  $W_1$  due to  $W_1$  be  $W_1 \Delta_1$ ,
- „ of  $A$  in the direction of  $W_1$  due to  $W_2$  be  $W_2 \Delta'_1$ ,
- „ of  $B$  in the direction of  $W_2$  due to  $W_2$  be  $W_2 \Delta_2$
- and that of  $B$  in the direction of  $W_2$  due to  $W_1$  be  $W_1 \Delta'_2$ .

Then the total movement of  $A$  in the direction of  $W_1$  is

$(W_1\Delta_1 + W_2\Delta'_1)$ , and the total movement of B in the direction of  $W_2$  is  $(W_2\Delta_2 + W_1\Delta'_2)$ .

Hence the total work done by  $W_1$  and  $W_2$  is

$$U = \frac{1}{2}W_1\{W_1\Delta_1 + W_2\Delta'_1\} + \frac{1}{2}W_2\{W_2\Delta_2 + W_1\Delta'_2\}.$$

Now if  $W_1$  acts alone, the work it does is  $\frac{1}{2}W_1^2\Delta_1$ .

Without altering  $W_1$  allow  $W_2$  to come into operation. Then the work done by  $W_2$  at B is  $\frac{1}{2}W_2^2\Delta_2$ , and the work done by  $W_1$  owing to the introduction of  $W_2$  is  $W_1W_2\Delta'_1$ .

Therefore the total work done by  $W_1$  and  $W_2$  is

$$U = \frac{1}{2}W_1^2\Delta_1 + \frac{1}{2}W_2^2\Delta_2 + W_1W_2\Delta'_1.$$

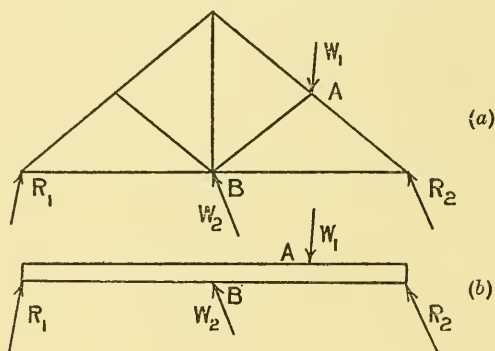


FIG. 14.

Equating these two values of  $U$  we get

$$\frac{1}{2}W_1\{W_1\Delta_1 + W_2\Delta'_1\} + \frac{1}{2}W_2\{W_2\Delta_2 + W_1\Delta'_2\} = \frac{1}{2}W_1^2\Delta_1 + \frac{1}{2}W_2^2\Delta_2 + W_1W_2\Delta'_1$$

$$\text{so } \frac{1}{2}W_1W_2\Delta'_1 + \frac{1}{2}W_1W_2\Delta'_2 = W_1W_2\Delta'_1$$

$$\text{or } \Delta'_2 = \Delta'_1.$$

That is, the deflection at B in the direction of  $W_2$ , when a unit load acts at A in the direction of  $W_1$ , is the same as the deflection at A in the direction of  $W_1$  when a unit load acts at B in the direction of  $W_2$ .

As an example, suppose that a weight  $W$  is carried at the end of a cantilever of span  $L$  as shown in Fig. 15, and the deflection at the mid-point is required.

By the usual methods we have

$$EI \frac{d^2y}{dx^2} = Wx,$$

$$EI \frac{dy}{dx} = \frac{Wx^2}{2} + A.$$

When  $x=L$ ,  $\frac{dy}{dx}=0$  and so  $A=-\frac{WL^2}{2}$ .

Therefore  $EIy=\frac{Wx^3}{6}-\frac{WL^2x}{2}+B$ .

When  $x=L$ ,  $y=0$  and so  $B=\frac{WL^3}{3}$ .

Therefore  $EIy=\frac{Wx^3}{6}-\frac{WL^2x}{2}+\frac{WL^3}{3}$ .

When  $x=\frac{L}{2}$ ,  $y_B=\frac{5WL^3}{48EI}$ .

The same result can be obtained by an application of Clerk Maxwell's theorem as follows:—

The deflection of B under a load W at A = the deflection of A under a load W at B.

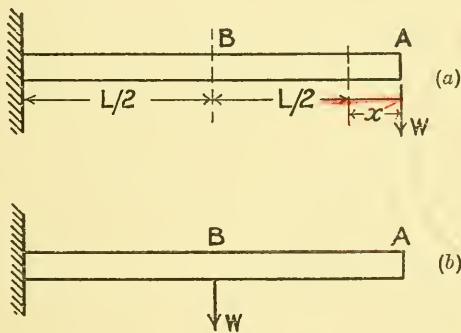


FIG. 15.

Now the deflection of A under a load W at B (Fig. 15 (b)) is

deflection at B +  $\frac{L}{2}$  (slope of beam at B)

or 
$$\frac{WL^3}{24EI} + \frac{WL^3}{16EI}$$

and 
$$y_B = \frac{5WL^3}{48EI}$$

as obtained by the first method. It often happens that an application of this theorem will save a considerable amount of laborious calculation.

**First Theorem of Castigliano.**—By the help of the theorem proved in the last paragraph we can obtain an exceedingly valuable result which was due in the first place to Castigliano.

## 26 STRAIN ENERGY METHODS OF STRESS ANALYSIS

Let any frame or elastic solid be in equilibrium under the action of a number of forces  $W_1, W_2, W_3, \dots W_N$ , which are supposed to act at the points 1, 2, 3  $\dots$  N. Let the

movement of point 1 in direction of  $W_1$  due to  $W_1$  be  $W_1(1\delta_1)$ .  
 that of point 1 in direction of  $W_1$  due to  $W_2$  be  $W_2(2\delta_1)$ .  
 that of point 1 in direction of  $W_1$  due to  $W_N$  be  $W_N(N\delta_1)$ .  
 that of point 2 in direction of  $W_2$  due to  $W_1$  be  $W_1(1\delta_2)$ , etc.

Then the total movement of  $W_1$  in its own line of action is

$$\Delta_1 = W_1(1\delta_1) + W_2(2\delta_1) + W_3(3\delta_1) + \dots + W_N(N\delta_1),$$

and the total movement of  $W_2$  in its own line of action is

$$\Delta_2 = W_1(1\delta_2) + W_2(2\delta_2) + W_3(3\delta_2) + \dots + \bar{W}_N(N\delta_2), \text{ etc.}$$

Then the total work done by the external forces being equal to the strain energy, we get

$$\begin{aligned} & \frac{1}{2} W_1[W_1(1\delta_1) + \dots + \bar{W}_N(N\delta_1)] + \frac{1}{2} W_2[W_1(1\delta_2) + \dots + \bar{W}_N(N\delta_2)] + \\ & \dots + \frac{1}{2} W_N[W_1(1\delta_N) + \dots + \bar{W}_N(N\delta_N)] \\ & = \frac{1}{2} \sum \frac{(\alpha W_1 + \beta W_2 + \gamma W_3 + \dots + \nu W_N)^2 L}{AE} \end{aligned}$$

where  $(\alpha W_1 + \beta W_2 + \dots + \nu W_N)$  is the load in any member due to the external forces.

Consider now the same body under the action of  $W_2, W_3, \dots W_N$ , *i.e.* with  $W_1$  removed.

We have for the movements of points 1, 2, etc., the expressions

$$\begin{aligned} & W_2(2\delta_1) + W_3(3\delta_1) + \dots + W_N(N\delta_1), \\ & W_2(2\delta_2) + W_3(3\delta_2) + \dots + \bar{W}_N(N\delta_2), \text{ etc.} \end{aligned}$$

And equating internal and external work as before :

$$\begin{aligned} & \frac{1}{2} W_2[W_2(2\delta_2) + \dots + \bar{W}_N(N\delta_2)] + \frac{1}{2} W_3[W_2(2\delta_3) + \dots + \bar{W}_N(N\delta_3)] + \\ & \dots + \frac{1}{2} W_N[W_2(2\delta_N) + \dots + \bar{W}_N(N\delta_N)] \\ & = \frac{1}{2} \sum \frac{(\beta W_2 + \gamma W_3 + \dots + \nu W_N)^2 L}{AE} \end{aligned}$$

Subtracting this equation from the previous one where all loads are acting, we get

$$\begin{aligned} & \frac{1}{2} W_1[W_1(1\delta_1) + W_2(2\delta_1) + \dots + W_N(N\delta_1)] + \frac{1}{2} W_2 W_1(1\delta_2) + \frac{1}{2} W_3 W_1(1\delta_3) \\ & \dots + \frac{1}{2} W_N W_1(1\delta_N) \\ & = \frac{1}{2} \sum \frac{\alpha W_1 L}{AE} [\alpha W_1 + 2\beta W_2 + 2\gamma W_3 + \dots + 2\nu \bar{W}_N]. \end{aligned}$$

By Clerk Maxwell's reciprocal theorem we can write

$$2\delta_1 = 1\delta_2, \quad 3\delta_1 = 1\delta_3, \quad \dots \quad N\delta_1 = 1\delta_N.$$

Hence

$$\begin{aligned} \frac{1}{2} W_1^2(\delta_1) + W_1[W_2(\delta_1) + \bar{W}_3(\delta_1) + \dots + \bar{W}_N(\delta_1)] \\ = \frac{1}{2} \sum \frac{L\alpha W_1}{AE} [(\alpha W_1 + 2\beta W_2 + 2\gamma W_3 + \dots + 2\nu W_N)]. \end{aligned}$$

If  $W_1$  acts alone on the body

$$\frac{1}{2} W_1\{W_1(\delta_1)\} = \frac{1}{2} \sum \frac{\alpha^2 W_1^2 L}{AE}.$$

And adding this to the last result we obtain

$$\begin{aligned} W_1[W_1(\delta_1) + W_2(\delta_1) + \dots + \bar{W}_N(\delta_1)] \\ = \sum \frac{L\alpha W_1}{AE} [\alpha W_1 + \beta W_2 + \dots + \nu \bar{W}_N]. \end{aligned}$$

or

$$W_1 \Delta_1 = W_1 \sum \frac{\partial u}{\partial W_1},$$

where  $u$  is the strain energy of an element of the body, *i.e.* of one bar in the case of the frame.

Hence

$$\Delta_1 = \frac{\partial U}{\partial W_1} \quad . \quad . \quad . \quad . \quad . \quad (10)$$

where  $U$  is the total strain energy of the body.

*That is, the partial differential coefficient of the total strain energy expressed in terms of the external load system, with respect to one of*

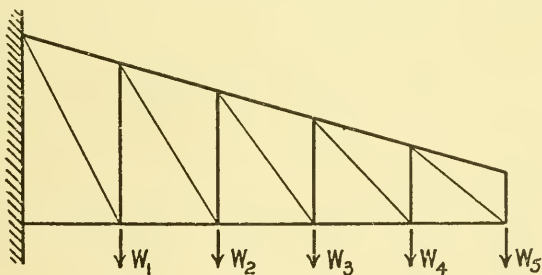


FIG. 16.

*the external loads, is the movement of that load in its line of action.* This is the first theorem of Castigliano.

As an example of the operation of this theorem consider the frame shown in Fig. 16.

By means of a stress diagram or otherwise, find all the forces in the bars of the frame. The force in any member will be of the form

$$P_0 = \{\alpha W_1 + \beta W_2 + \gamma W_3 + \delta W_4 + \epsilon W_5\},$$

where  $\alpha, \beta, \gamma, \delta$  and  $\epsilon$  are numerical coefficients, varying for each member, and in some cases being zero.

Then the strain energy in one member is

$$u = \frac{1}{2} \frac{P_0^2 L}{AE},$$

where  $L$  is the length of the bar and  $A$  is its cross-sectional area. The total strain energy is

$$U = \Sigma u = \Sigma \frac{1}{2} \frac{P_0^2 L}{AE}$$

where the summation extends to all the members of the frame.

Then by the first theorem,

$$\frac{\partial u}{\partial W_1} = \text{movement of } W_1 \text{ in its own line of action.}$$

$$\frac{\partial u}{\partial W_2} = \text{movement of } W_2 \text{ in its own line of action, and so on.}$$

Since we only need the differential coefficient of  $U$  and not  $U$  itself, it is not necessary to calculate the total strain energy, which, as it involves the squaring of the terms for  $P_0$ , is often a laborious process. Instead, we can carry out the differentiation at an earlier stage as follows :—

$$U = \frac{1}{2} \Sigma \frac{P_0^2 L}{AE}$$

$$\therefore \frac{\partial U}{\partial W_1} = \frac{1}{2} \Sigma \frac{\partial}{\partial W_1} \left( \frac{P_0^2 L}{AE} \right) = \Sigma \frac{P_0 L}{AE} \frac{\partial P_0}{\partial W_1}.$$

$$\text{But } \frac{\partial P_0}{\partial W_1} = \alpha$$

$$\text{and so } \frac{\partial U}{\partial W_1} = \Sigma \frac{L\alpha}{AE} \{ \alpha W_1 + \beta W_2 + \gamma W_3 + \delta W_4 + \epsilon W_5 \},$$

$$\frac{\partial U}{\partial W_2} = \Sigma \frac{L\beta}{AE} \{ \alpha W_1 + \beta W_2 + \gamma W_3 + \delta W_4 + \epsilon W_5 \}, \text{ etc.}$$

In an actual problem the loads  $W_1, W_2, \dots, W_N$ , have numerical values, but as they must be kept distinct for purposes of differentiation, they should always be given symbols, the numerical values being put in as the final operation.

This entails the drawing of a separate stress diagram for each load in order that the coefficients  $\alpha, \beta$ , etc., may be found.

**Movement of an Unloaded Point in a Body.**—In many cases it is required to find the movement of a point in a body which is

unloaded, and the method is as follows : at the point in question imagine a load, say,  $W_x$ , and proceed to find the movement of  $W_x$  in the usual way. After differentiating as already described, put  $W_x$  equal to zero and the required result will be obtained. This is sometimes known as Frenkel's theorem.

**Second Theorem of Castigliano.**—The foregoing results will now be used to deduce the second theorem of Castigliano, which provides a method for the analysis of frames with redundant bars and of certain cases of elastic solids.

In Fig. 17 let A and B be two adjacent nodes of a frame subjected to external loads.

Let  $L$  be the original distance between A and B, before any loads are applied, loads in this case including any due to the forcing into position of redundant bars not of the correct initial length.

Let loads  $P$  and  $Q$  be applied to the points A and B as shown.

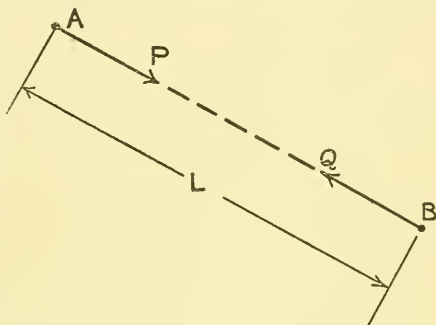


FIG. 17.

Then if  $U'$  is the total strain energy of the frame due to the action of all the external loads including  $P$  and  $Q$ , and all self-straining loads, we have by the first theorem of Castigliano :

$$\frac{\partial U'}{\partial P} = \text{movement of A in the direction of P.}$$

$$\frac{\partial U'}{\partial Q} = \text{movement of B in the direction of Q.}$$

If  $P=Q=R$  (say),

$$\frac{\partial U'}{\partial R} = \frac{\partial U'}{\partial P} + \frac{\partial U'}{\partial Q} = \text{shortening of distance AB.}$$

Suppose that the load  $R$  is applied by a member connecting A and B, *i.e.*  $R$  is the load in a redundant member of the frame.

Also, let the original length of this member be  $(L-\lambda)$ , where  $\lambda$  is small compared with  $L$ .

Then the final length of the member is  $(L-\lambda)\left(1+\frac{R}{AE}\right)$ , where

$R$  is assumed to be a tension.

Since A and B were originally separated by a distance L, they have approached by a distance  $L - (L - \lambda)\left(1 + \frac{R}{AE}\right)$ .

Neglecting the second order term  $\frac{\lambda R}{AE}$ , the amount of their approach is  $\lambda - \frac{LR}{AE}$ .

Hence 
$$\lambda - \frac{LR}{AE} = \frac{\partial U'}{\partial R},$$

or 
$$\frac{\partial U'}{\partial R} + \frac{LR}{AE} = \lambda.$$

But  $\frac{LR}{AE} = \frac{\partial u}{\partial R}$ , where  $u$  is the strain energy of the member AB. Therefore if  $U = U' + u$  = the total strain energy of the frame including the bar AB, we have

$$\frac{\partial U}{\partial R} = \lambda \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

That is, *the partial differential coefficient of the total strain energy with respect to the load in a redundant member is equal to the initial lack of fit of that member.*

If R had been compressive, the approach of A to B would have been  $\left(\lambda + \frac{RL}{AE}\right)$ , and in this case,  $\frac{\partial U'}{\partial R}$  being measured in the opposite direction to the force, would be negative, so that we should have

$$\lambda + \frac{LR}{AE} = - \frac{\partial U'}{\partial R},$$

or 
$$\frac{\partial U}{\partial R} = -\lambda.$$

The negative sign arises from the fact that a member initially too short is assumed to have a final compression in it. In order to keep the signs correct it should always be assumed that if a member is too short the final load in it is tensile, and if too long that it is compressive. If this rule is observed,  $\frac{\partial U}{\partial R} = \lambda$ , and the final result will show whether R is tensile or compressive by the sign attached to it.

This is the second theorem of Castigliano in its general form. If the member AB is originally of the exact length required,  $\lambda = 0$ , and we get

$$\frac{\partial U}{\partial R} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

This is generally known as the principle of least work, since it is the condition that the value of  $R$  shall cause the strain energy in the frame to be a minimum. This so-called principle is, however, as has been shown, only a particular case of a more general theorem.

Suppose the frame shown in Fig. 18 is to be analysed, the procedure is as follows:—

In the first place it is evident that there are two redundant members. Any two may be considered as redundant provided that the remaining bars form a complete frame, and so we may take  $CE$  and  $BD$ .

It will be assumed that  $CE$  was initially too short by an amount  $\lambda_1$  and  $BD$  too long by an amount  $\lambda_2$ .

Replace  $CE$  and  $BD$  by forces  $R_1$  and  $R_2$  acting on the joints  $C$  and  $E$ , and  $B$  and  $D$  respectively as shown at (b) in the figure. By stress diagrams or any other means the forces in all the other members of the frame must be found in terms of the unknowns  $R_1$  and  $R_2$ , and the external forces  $W_1$  and  $W_2$ . It should be noted that, if the graphical method is used, it will be necessary to draw three diagrams, one each for the forces  $R_1$  and  $R_2$ , and one for the external load system.

Thus the loads in all the members of the frame are obtained in the form

$$P_0 = (aW_1 + \beta W_2 + aR_1 + bR_2),$$

or if  $W_1$  and  $W_2$  are numerical values, this can be written as

$$P_0 = (A + aR_1 + bR_2),$$

where  $A$  is in force units and  $a$  and  $b$  are numerical coefficients.

The total strain energy of the frame is

$$U = \left( \sum \frac{P_0^2 L}{AE} \right) + \frac{1}{2} \frac{R_1^2 L_1}{A_1 E} + \frac{1}{2} \frac{R_2^2 L_2}{A_2 E},$$

the summation indicating all bars of the just-stiff frame, or the essential bars as they are often called, and the remaining terms relating to the two redundant members.

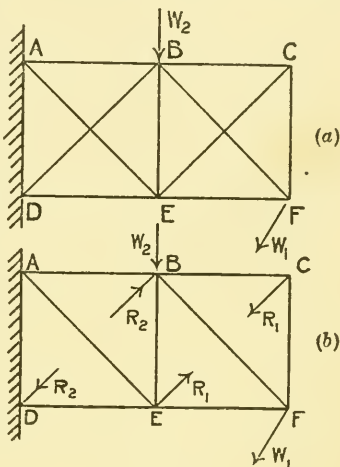


FIG. 18.

Then 
$$\frac{\partial U}{\partial R_1} = \left( \Sigma a \frac{P_0 L}{AE} \right) + \frac{R_1 L_1}{A_1 E} = \lambda_1,$$

and 
$$\frac{\partial U}{\partial R_2} = \left( \Sigma b \frac{P_0 L}{AE} \right) + \frac{R_2 L_2}{A_2 E} = \lambda_2.$$

Thus there are two equations to be solved for  $R_1$  and  $R_2$ .

The same procedure is followed for any number of redundant members, and in any case where these were initially of the correct length,  $\lambda$  is put equal to zero. A fully worked example of the application of this method is given in Part II, Example 1, where a way of tabulating the calculations is given in detail.

**Differential Coefficients of Strain Energy with Respect to a Moment.**—In the previous paragraphs it has been assumed that the loads were all axial, and no mention has been made of bending

moments acting on an elastic body.

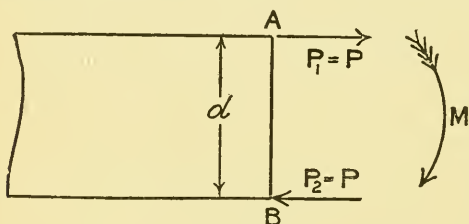


FIG. 19.

In Fig. 19, let a bending moment  $M$  act at the section  $AB$  of an elastic body. Whether this bending moment is an internal redundancy or an external action, it may be supposed to

consist of two equal and opposite forces of magnitude  $P$ , acting as shown in the figure, so that if the lever arm is  $d$  we have  $Pd = M$ .

If now we distinguish these forces by calling one of them  $P_1$ , and the other  $P_2$ , so that  $P_1$  acts at  $A$  and  $P_2$  acts at  $B$ , we have

$$\frac{\partial U}{\partial P_1} + \frac{\partial U}{\partial P_2} = \frac{\partial U}{\partial P}.$$

If  $M$  is an external action,  $P_1$  and  $P_2$  are external forces, and

$$\frac{\partial U}{\partial P_1} = \delta_1 = \text{movement of } P_1,$$

and 
$$\frac{\partial U}{\partial P_2} = \delta_2 = \text{movement of } P_2.$$

Therefore 
$$\frac{\partial U}{\partial P} = (\delta_1 + \delta_2)$$

and 
$$\frac{1}{d} \cdot \frac{\partial U}{\partial P} = \frac{1}{d} (\delta_1 + \delta_2)$$

but 
$$\frac{1}{d} \cdot \frac{\partial U}{\partial P} = \frac{\partial U}{\partial (Pd)} = \frac{\partial U}{\partial M}.$$

Also  $\frac{1}{d}(\delta_1 + \delta_2) = \theta$  = the angular rotation of AB, since on the ordinary assumptions of bending, AB remains plane.

$$\text{Hence} \quad \frac{\partial U}{\partial M} = \theta \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

If M is a redundant internal action,  $P_1$  and  $P_2$  may be considered as redundant forces, so that

$$\frac{\partial U}{\partial P_1} = \frac{\partial U}{\partial P_2} = 0.$$

Therefore  $\frac{\partial U}{\partial M} = 0$ , and both the theorems of Castigliano are seen to be applicable.

**Temperature Stresses.**—The theorems of the preceding paragraphs can be used to solve problems connected with the effect of temperature changes on certain structures. Suppose, for example, a steel roof truss to be fixed at one end and mounted on a roller bearing at the other. When the roof experiences a rise in temperature every member will extend slightly, and since one end of the truss is free to move, and the extensions of all members are proportional to their lengths, there will be no effect other than a small change of scale in the structure. If, however, instead of being free to move the truss is attached to rigid supports, the effect of a temperature change will be to cause internal forces in the members of the truss, since the case is now exactly equivalent to that of forcing a structure on to two supports which are not at the correct distance apart. In the same way, if one member of a redundant truss becomes subjected to a change of temperature relative to the remainder of the structure, the position is equivalent to that of forcing a member too short or too long into position.

Suppose, then, that one member of a redundant frame becomes heated to a temperature  $t^\circ$  above the other bars of the frame.

Let the original length of this member be  $L$ .

If the member be disconnected from the truss before heating, its new length is  $L(1 + at)$ , where  $a$  is the coefficient of expansion of the material. If the heated member be now replaced, it is too long by the amount  $Lat$ , and has to be forced into position, which produces a compressive load in it of unknown amount  $T$ . This load will stress the remainder of the frame,

and the strain energy of the truss can be found in the ordinary way in terms of  $T$ . Then,

$$\frac{\partial U}{\partial T} = L\alpha t$$

which provides the equation to determine the value of  $T$ .

If the frame is redundant, and is not made of the same material throughout, a uniform change of temperature of the frame in general causes different proportional expansions in the different members, since the coefficients of expansion for the materials are not the same. In this case temperature stresses are set up which can be calculated as follows:—

In the first place the frame must be reduced to a just-stiff structure by the removal of all redundant bars. In selecting redundancies it is advisable to arrange, if possible, that all the essential bars of the frame are made of the same material, as this simplifies the calculations. This just-stiff frame is now supposed to be heated to the temperature specified by the conditions of the problem, and the distances between the points to be connected by the removed redundancies are calculated. If all the essential bars are of the same material, this is a simple matter, since the distance between any two points originally at a distance  $L$  apart will be  $(1+\alpha t)L$ . If the essential bars are of different materials the frame in its heated state is distorted, due to the different proportional expansions of the members, and the new distances must be found by a deflection diagram or similar means.

The removed bars are now supposed to be heated through the same rise of temperature, and their new lengths found.

Suppose that the distance between two points  $A$  and  $B$  of the simple frame has increased by an amount  $L\alpha t$ , due to the rise in temperature, and the redundant bar  $AB$  which has to connect these points has increased in length by an amount  $L\alpha_1 t$ , due to the same rise in temperature, then the bar  $AB$  on replacement is too short by the amount  $L(\alpha - \alpha_1)t$ , and the equation obtained by differentiating with respect to the redundancy  $R_{AB}$  is

$$\frac{\partial U}{\partial R_{AB}} = L(\alpha - \alpha_1)t.$$

In the same way equations are found for all the other redundant bars, and their simultaneous solution gives the loads in the structure due to the rise in temperature.

As an example of the treatment of these temperature stresses, we will take the case of a flywheel on a prime mover, which is

used as a brake wheel. The frictional forces on the rim cause a rise in the temperature of the rim. This in general is not communicated to the arms, which can be assumed to remain at their original temperature.

Imagine the arms disconnected from the rim which is then free to expand. If  $R_1$ , Fig. 20, was the original radius to the inner face of the rim and  $t$  the rise in temperature, the increase in radius will be  $R_1 \alpha t$ .

If the arms are now reconnected to the rim, a force  $T$  is applied to the rim at each point of attachment, acting radially inwards, since the arms are in tension.

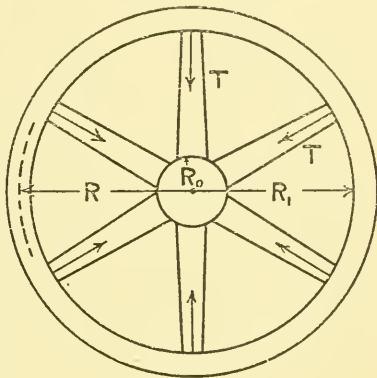


FIG. 20.

From the symmetry of the wheel it is evident that the conditions in each segment of the rim must be the same, so we need only consider one arm and one rim segment.

Let  $U$  be the strain energy of one arm and rim segment due to the force  $T$ . Then, in Part II, Example 6, it is shown that

$$\frac{dU}{dT} = \frac{T}{E} \left[ (\theta \operatorname{cosec}^2 \theta + \cot \theta) \left( \frac{R^3}{4I} + \frac{R}{4A} \right) - \frac{R^3}{2I\theta} + \frac{l}{a_0} \right]$$

where  $2\theta$  is the angle between adjacent arms,

$R$  is the radius to the centroid of the section of rim,

$I$  is the moment of inertia of the cross-section of the rim,

$A$  is the cross-sectional area of the rim,

$l$  is the length of an arm  $= (R_1 - R_0)$ ,

$a_0$  is the cross-sectional area of an arm.

Then, since the arm was too short by an amount  $R_1 \alpha t$ ,

$$\frac{T}{E} \left[ (\theta \operatorname{cosec}^2 \theta + \cot \theta) \left( \frac{R^3}{4I} + \frac{R}{4A} \right) - \frac{R^3}{2I\theta} + \frac{l}{a_0} \right] = R_1 \alpha t$$

$$\text{or, } T = \frac{ER_1 \alpha t}{\left[ (\theta \operatorname{cosec}^2 \theta + \cot \theta) \left( \frac{R^3}{4I} + \frac{R}{4A} \right) - \frac{R^3}{2I\theta} + \frac{l}{a_0} \right]}$$

This tension may be very serious and several cases have occurred of the failure of flywheels due to this cause.

A procedure similar to the above will solve most problems connected with stresses due to temperature variation.

## CHAPTER III

### BEAMS, CURVED MEMBERS, AND STIFF-JOINTED FRAMES

**Essential and Redundant Reactions in a Beam.**—It has already been seen that in the general case a plane frame needs three reactive forces and a space frame six. More than these introduce redundant reactions.

The same is true of solid bodies: the reactions must satisfy the laws of static equilibrium. In the case of plane structures this needs forces parallel to two perpendicular axes and also a couple to balance the applied loads. Similarly for a body in space the reactions must provide the six essential elements—three forces and three moments about mutually perpendicular axes.

In many cases of beams the loads are perpendicular to the axis of the beam and so there are no axial components to be balanced. A pin is then not essential, and complete reactions are provided by simply resting the beam on two supports. It must be remembered, however, that this is really a special case of loading, and the conditions for complete support demand that the body shall be capable of supporting *any* applied load system.

A cantilever just fulfils the essential conditions, since at the point where it is held it can exert a force along the axis of the beam, a shear at right angles to this axis and a couple which will balance the applied moments.

If, however, an additional support is provided, as in Fig. 21,

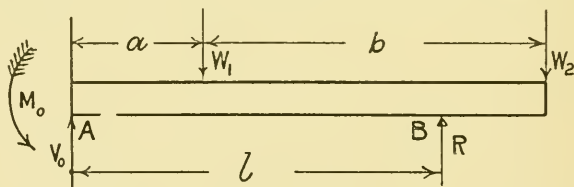


FIG. 21.

there is one redundancy, and the beam may be treated by strain energy methods.

Alternative treatments are available, *e.g.* we may imagine the beam to be simply supported at A and B and loaded with  $W_1$  and  $W_2$  and the moment  $M_0$  at A. This bending moment is then a redundancy which must be evaluated.

The reactions at A and B can be determined in terms of the loads and  $M_0$ , by taking moments about A and equating vertical forces.

$$\text{Thus} \quad M_0 + Rl = aW_1 + (a+b)W_2$$

$$\text{or} \quad R = \frac{aW_1 + (a+b)W_2 - M_0}{l},$$

$$\text{and} \quad V_0 = (W_1 + W_2) - R,$$

the bending moment at any point can now be expressed in terms of the known forces and the unknown  $M_0$ .

Since the end A is firmly built into the wall there can be no angular movement of the wall section of the beam, which is expressed by the statement

$$\frac{dU}{dM_0} = 0.$$

$$\text{And since} \quad \frac{dU}{dM_0} = \frac{1}{EI} \int M_x \cdot \frac{dM_x}{dM_0} \cdot dx,$$

it is only necessary to evaluate the integral to find the value of  $M_0$ .

A second, and in this case, a simpler method is to treat the reaction  $R$  as an extra external force on the beam, so that we have a cantilever loaded with known forces  $W_1$  and  $W_2$  and an unknown force  $R$ .

If  $U$  is the total strain energy of the beam in terms of  $R$ , etc., the deflection of  $R$  is given by the relation

$$\frac{dU}{dR} = \delta.$$

But since  $B$  is a fixed point there is no movement of  $R$ , and so

$$\frac{dU}{dR} = 0,$$

which provides the necessary equation to solve for  $R$ .

**Beam Resting on more than Two Supports.**—The case of a beam resting on more than two supports is dealt with in the same way.

In Fig. 22, for example, a beam ABCD is shown resting on four supports, which are fixed. Loads  $W_1$  and  $W_2$  are carried

as shown. Since only two supports are necessary, the other two may be treated as redundant. As in the previous case it is the better plan to view the beam as being loaded with four loads, two of which,  $W_1$  and  $W_2$ , are known, and two are such that they do not move when the beam is loaded. If we take  $R_2$  and  $R_3$  as

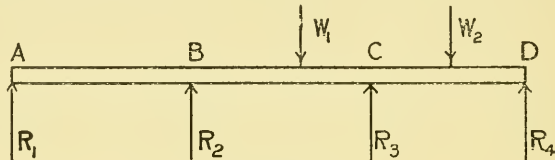


FIG. 22.

the two forces to be determined, we can find  $R_1$  and  $R_4$  in terms of these and  $W_1$  and  $W_2$ .

The bending moment at any point in the beam can then be expressed in terms of the external loads  $W_1$ ,  $W_2$  and  $R_2$ ,  $R_3$ .

Since there is no movement of  $R_2$  and  $R_3$ , we have

$$\frac{\partial U}{\partial R_2} = \text{movement of } R_2 \text{ in its line of action} = 0.$$

$$\frac{\partial U}{\partial R_3} = \text{movement of } R_3 \text{ in its line of action} = 0.$$

Hence 
$$\frac{\partial U}{\partial R_2} = \frac{1}{EI} \int M_x \frac{\partial M_x}{\partial R_2} dx = 0$$

and 
$$\frac{\partial U}{\partial R_3} = \frac{1}{EI} \int M_x \frac{\partial M_x}{\partial R_3} dx = 0,$$

which provide the two equations for the evaluation of  $R_2$  and  $R_3$ .

As the bending moment changes its form, the integration will have to be done in steps, *i.e.* between  $R_1$  and  $R_2$ ,  $R_2$  and  $W_1$ , etc.

**Beams on Sinking Supports and Propped Beams.**—Suppose that, instead of the points B and C being fixed, it is known that under load they sink by specified amounts  $\delta_1$  and  $\delta_2$ .

Then  $\frac{\partial U}{\partial R_2}$  is no longer zero, but  $-\delta_2$ , and  $\frac{\partial U}{\partial R_3}$  is  $-\delta_3$ ; the negative sign indicating that the movement of the points is in the opposite direction to the forces.

On substituting the known values of  $\delta_1$  and  $\delta_2$  instead of zero, the new values of  $R_2$  and  $R_3$  are found.

A similar problem arises in the case where a beam is propped by supports. Suppose, for example, that the beam in Fig. 22 is propped at B and C by known amounts  $\lambda_2$  and  $\lambda_3$ . Then, if

the free deflections at B and C are  $\Delta_2$  and  $\Delta_3$ , the final positions of B and C are  $(\Delta_2 - \lambda_2)$  and  $(\Delta_3 - \lambda_3)$  below the line AD.

$$\text{Then} \quad \frac{\partial U}{\partial R_2} = -(\Delta_2 - \lambda_2),$$

$$\text{and} \quad \frac{\partial U}{\partial R_3} = -(\Delta_3 - \lambda_3).$$

Having found  $\frac{\partial U}{\partial R_2}$  and  $\frac{\partial U}{\partial R_3}$ , the values of  $\Delta_2$  and  $\Delta_3$  are first found by putting  $R_2$  and  $R_3$  equal to zero in the resulting expressions,  $(\Delta_2 - \lambda_2)$  and  $(\Delta_3 - \lambda_3)$  are then known, and the above equations are easily solved.

**Relative Importance of Bending, Axial Loading, and Shear Loading Terms in the Strain Energy Account.**—Generally speaking, in all problems dealing with curved bars, links, or cranked bars, the resultant actions at any section consist of a bending moment, an axial tension or compression, and a transverse shearing force. All of these should, strictly, be taken into the strain energy account, but in practically every case the shearing effect is so small that it can quite safely be ignored. In many cases also the term due to direct load is sufficiently small to be neglected without introducing a serious error. From the nature of the problem it is not possible to give a general proof of this statement, but the following example will help to show that it is reasonable.

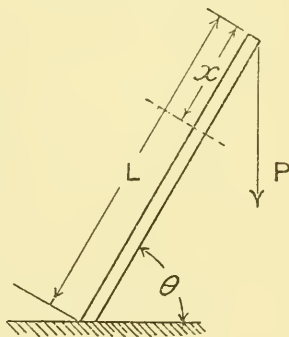


FIG. 23.

Suppose that a bar of length  $L$ , inclined at an angle  $\theta$  to the horizontal, carries a load  $P$  acting vertically at its free end, as shown in Fig. 23.

At any section a distance  $x$  from the free end measured along the bar, the load exerts resultant actions consisting of :

$$\begin{aligned} \text{an axial compressive force} &= P \sin \theta \\ \text{a transverse shearing force} &= P \cos \theta \\ \text{a bending moment} &= xP \cos \theta \end{aligned}$$

The total strain energy is the sum of the terms due to these three resultant actions.

The energy due to bending is

$$U_B = \frac{1}{2EI} \int_0^L (xP \cos \theta)^2 dx = \frac{P^2 L^3 \cos^2 \theta}{6EI},$$

due to the axial force it is

$$U_D = \frac{1}{2AE} \int_0^L (P \sin \theta)^2 dx = \frac{P^2 L \sin^2 \theta}{2AE},$$

and due to shear it is

$$U_S = \frac{1}{2AN} \int_0^L (P \cos \theta)^2 dx = \frac{P^2 L \cos^2 \theta}{2AN};$$

if it be assumed that the shear force causes a uniform stress over the cross-section of the bar.

$$\text{Hence } U_B : U_D : U_S = \frac{L^2 \cos^2 \theta}{3EI} : \frac{\sin^2 \theta}{AE} : \frac{\cos^2 \theta}{AN}.$$

Consider first the relative magnitude of the terms due to bending and shear. If we put  $N = \frac{2}{5}E$  we get :

$$\frac{U_B}{U_S} = \frac{L^2 AN}{3EI} = \frac{2L^2}{15k^2},$$

$k$  being the radius of gyration of the section. Unless the member is extremely short  $\left(\frac{L}{k}\right)^2$  is large and  $U_S$  is negligible compared with  $U_B$ .

Now consider the relative importance of the bending and axial loading terms. We have :

$$\frac{U_B}{U_D} = \frac{L^2 \cot^2 \theta}{3k^2}.$$

This depends not only on the ratio  $\left(\frac{L}{k}\right)$  but also on the angle  $\theta$ ; when the bar is vertical the load is wholly axial and  $U_B$  is zero.

Suppose that the bar is square in cross-section, having a side  $s$  :—

$$\text{Then } \frac{U_B}{U_D} = 4 \left(\frac{L}{s}\right)^2 \cot^2 \theta.$$

Even in the case of a short bar where  $\frac{L}{s}$  is 10, the deviation from the vertical needs to be less than  $3^\circ$  for the bending strain energy to be equal to that due to compression, and if  $\theta$  is  $45^\circ$  the latter is only 0.25 per cent. of the former.

The terms generally required, however, are not the actual values of the strain energies, but the differential coefficients with respect to the load.

In the foregoing example, for instance, the vertical movement of the point carrying the load is given by

$$\begin{aligned}\frac{dU}{dP} &= \frac{d}{dP} \left[ \frac{P^2 L^3 \cos^2 \theta}{6EI} + \frac{P^2 L \sin^2 \theta}{2AE} + \frac{P^2 L \cos^2 \theta}{2AN} \right] \\ &= 2P \left[ \frac{L^3 \cos^2 \theta}{6EI} + \frac{L \sin^2 \theta}{2AE} + \frac{L \cos^2 \theta}{2AN} \right],\end{aligned}$$

and the ratios of the movements due to bending, axial compression, and shear are the same as the ratios of the strain energies found previously.

It is clear, therefore, that in many cases the terms due to shear and axial forces can be neglected in comparison with those due to bending.

**The Bow Girder.**—On account of the torsional effects called into play, the bow girder, or girder which is curved in plan form, presents an interesting problem in stress analysis. A girder of this type must be firmly fixed at each end and the reactions then consist of a bending moment, a shear, and a torque. Similarly the resultant action at every section consists of three corresponding terms.

An analysis for the case of the circular-arc bow girder has been given by Gibson and Ritchie<sup>1</sup> based upon a consideration of the slope changes; and the formulæ obtained can be applied to any girder whose plan form is composed of circular arcs.

The method<sup>2</sup> based on strain energy can be used for a girder of any plan form, and although in many cases the solution entails a considerable amount of graphical work, it presents no inherent difficulty. In the special case of the circular arc the results obtained agree with those of the work previously cited.

Let Fig. 24 represent a bow girder firmly built into walls or otherwise completely restrained at A and B.

Let a load W act vertically at C. Then at any section X of the girder the resultant actions consist of:

- a bending moment  $M_x$  with its axis normal to the curve at X,
- a vertical shearing force  $F_x$ ,
- a torque  $T_x$  with its axis tangential to the curve at X.

<sup>1</sup> *A Study of the Circular-Arc Bow Girder.* Gibson and Ritchie. Constable.

<sup>2</sup> *The Stress Analysis of Bow Girders.* A. J. S. Pippard and F. L. Barrow. H.M. Stationery Office, 1926.

If we imagine the girder to be severed at B across a plane normal to the curve at that point, we must, to restore the conditions of equilibrium previously existing, apply to the cut section an external bending moment  $M_0$ , a torque  $T_0$ , and a shearing force  $F_0$ .

These are provided by the wall and constitute the reactions at B. The resultant actions at the section X can now be expressed in terms of  $M_0$ ,  $T_0$ ,  $F_0$  and the load  $W$ , and to obtain a complete solution of the stress distribution in the girder it is only necessary to evaluate these reactions.

Since the girder is firmly built in at A and B, there is no

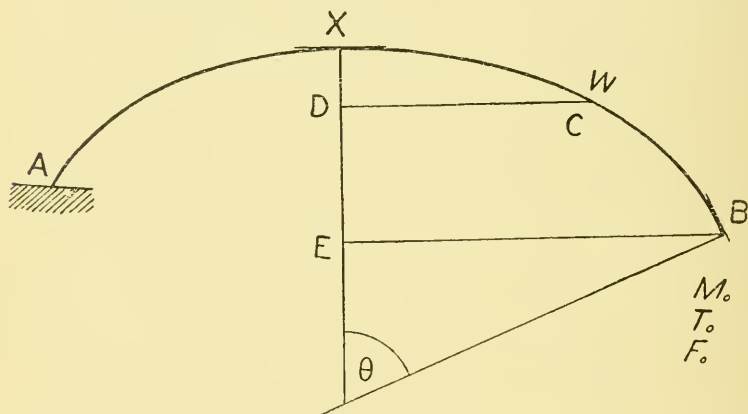


FIG. 24.

movement at either end, so if  $U$  is the total strain energy of the girder we can write :

$$\frac{\partial U}{\partial M_0} = \text{angular movement of B in direction of } M_0 = 0,$$

$$\frac{\partial U}{\partial T_0} = \text{angular movement of B in direction of } T_0 = 0,$$

$$\frac{\partial U}{\partial F_0} = \text{vertical movement of B} = 0.$$

These relations provide three equations for  $M_0$ ,  $T_0$ , and  $F_0$ . The total strain energy of the beam is made up of three components due to bending, torque, and shear respectively. Denoting these by  $U_B$ ,  $U_T$ , and  $U_F$  we have

$$U = U_B + U_T + U_F.$$

The last term, *i.e.* the energy due to shear strain, is, as already

indicated, very small compared with the other terms, and will be neglected. We then obtain

$$\left. \begin{aligned} \frac{\partial U_B}{\partial M_0} + \frac{\partial U_T}{\partial M_0} &= 0 \\ \frac{\partial U_B}{\partial T_0} + \frac{\partial U_T}{\partial T_0} &= 0 \\ \frac{\partial U_B}{\partial F_0} + \frac{\partial U_T}{\partial F_0} &= 0 \end{aligned} \right\} \dots \dots \dots (15)$$

If  $ds$  is a small element of the arc of the girder,  $EI$  its flexural rigidity and  $NJ$  its torsional rigidity,

$$U_B = \int \frac{M_x^2}{2EI} ds, \quad U_T = \int \frac{T_x^2}{2NJ} ds,$$

where the integration is taken round the whole girder from B to A. Now

$$\left. \begin{aligned} \frac{\partial U_B}{\partial M_0} &= \int \frac{M_x}{EI} \frac{\partial M_x}{\partial M_0} ds \\ \frac{\partial U_T}{\partial M_0} &= \int \frac{T_x}{NJ} \frac{\partial T_x}{\partial M_0} ds \\ \frac{\partial U_B}{\partial T_0} &= \int \frac{M_x}{EI} \frac{\partial M_x}{\partial T_0} ds \\ \frac{\partial U_T}{\partial T_0} &= \int \frac{T_x}{NJ} \frac{\partial T_x}{\partial T_0} ds \\ \frac{\partial U_B}{\partial F_0} &= \int \frac{M_x}{EI} \frac{\partial M_x}{\partial F_0} ds \\ \frac{\partial U_T}{\partial F_0} &= \int \frac{T_x}{NJ} \frac{\partial T_x}{\partial F_0} ds \end{aligned} \right\} \dots \dots \dots (16)$$

These values can be substituted in (15) to obtain the necessary three equations.

Before writing down expressions for  $M_x$ ,  $T_x$ , and  $F_x$  a convention as to signs must be adopted. Suppose the section at X of the part of the girder AX be viewed from the front, then it will be taken that,

- a positive moment at X is one that tends to make AX convex upwards,
- a positive torque at X is one that tends to turn the viewed section clockwise,
- a positive shear at X is one that tends to raise the viewed section.

With these conventions  $M_0$ ,  $T_0$ , and  $F_0$  are all positive, and at the section X

$$\begin{aligned} M_0 \text{ imposes a bending moment} & \quad . \quad = \quad M_0 \cos \theta \\ \text{and a torque} & \quad . \quad . \quad = -M_0 \sin \theta, \\ T_0 \text{ imposes a torque} & \quad . \quad . \quad = \quad T_0 \cos \theta \\ \text{and a bending moment} & \quad . \quad . \quad = \quad T_0 \sin \theta, \end{aligned}$$

where  $\theta$  is the angle between the normals to the curve of the girder at B and X

From B draw BE perpendicular to the normal at X and from C, the point of application of the load W, draw CD perpendicular to the same normal. Then if X lies between C and A we have

$$M_x = M_0 \cos \theta + T_0 \sin \theta - F_0 \cdot BE + W \cdot CD$$

$$T_x = -M_0 \sin \theta + T_0 \cos \theta + F_0 \cdot XE - W \cdot XD$$

If X lies between B and C, the expressions for  $M_x$  and  $T_x$  are the same with the terms in W omitted.

From the above equations for  $M_x$  and  $T_x$ ,

$$\frac{\partial M_x}{\partial M_0} = \cos \theta,$$

$$\frac{\partial T_x}{\partial M_0} = -\sin \theta,$$

$$\frac{\partial M_x}{\partial T_0} = \sin \theta,$$

$$\frac{\partial T_x}{\partial T_0} = \cos \theta,$$

$$\frac{\partial M_x}{\partial F_0} = -BE,$$

$$\frac{\partial T_x}{\partial F_0} = XE.$$

The values of  $M_x$ ,  $T_x$  and the differentials can now be substituted in equations (15) and (16). Putting  $\frac{1}{EI} = \alpha$  and  $\frac{1}{NJ} = \beta$  we obtain

$$\begin{aligned} M_0 \left[ \alpha \int_B^A \cos^2 \theta \, ds + \beta \int_B^A \sin^2 \theta \, ds \right] + \frac{T_0}{2} \left[ (\alpha - \beta) \int_B^A \sin 2\theta \, ds \right] \\ - F_0 \left[ \alpha \int_B^A BE \cos \theta \, ds + \beta \int_B^A XE \sin \theta \, ds \right] \\ + W \left[ \alpha \int_C^A CD \cos \theta \, ds + \beta \int_C^A XD \sin \theta \, ds \right] = 0. \quad . \quad . \quad (17) \end{aligned}$$

$$\begin{aligned}
 & \frac{M_0}{2} \left[ (\alpha - \beta) \int_B^A \sin 2\theta \, ds \right] + T_0 \left[ \alpha \int_B^A \sin^2 \theta \, ds + \beta \int_B^A \cos^2 \theta \, ds \right] \\
 & - F_0 \left[ \alpha \int_B^A BE \sin \theta \, ds - \beta \int_B^A XE \cos \theta \, ds \right] \\
 & + W \left[ \alpha \int_C^A CD \sin \theta \, ds - \beta \int_C^A XD \cos \theta \, ds \right] = 0. \quad \dots \quad (18) \\
 & - M_0 \left[ \alpha \int_B^A BE \cos \theta \, ds + \beta \int_B^A XE \sin \theta \, ds \right] \\
 & - T_0 \left[ \alpha \int_B^A BE \sin \theta \, ds - \beta \int_B^A XE \cos \theta \, ds \right] \\
 & + F_0 \left[ \alpha \int_B^A BE^2 ds + \beta \int_B^A XE^2 ds \right] \\
 & - W \left[ \alpha \int_C^A CD \cdot BE ds + \beta \int_C^A XD \cdot XE ds \right] = 0. \quad \dots \quad (19)
 \end{aligned}$$

It has been assumed that  $\alpha$  and  $\beta$  are constant. Modifications when these are variable round the curve will be noted later.

The terms involving  $M_0$ ,  $T_0$ , and  $F_0$  are obtained by integration round the whole curve, while those involving  $W$  require integration from the point of application of  $W$  to  $A$ . In the special case of a circular-arc bow girder the integrals in equations (17), (18), and (19) can be evaluated directly.

The first three terms in each equation are independent of the external load arrangement. If there is more than one load acting on the girder, each one requires coefficients similar to those of  $W$  in the above equations. Suppose, for example, loads  $W_1$ ,  $W_2 \dots W_n$  act at points on the girder denoted by 1, 2  $\dots n$ ; then in equation (17) the term

$$W \left[ \alpha \int_C^A CD \cos \theta \, ds + \beta \int_C^A XD \sin \theta \, ds \right]$$

becomes

$$\begin{aligned}
 & W_1 \left[ \alpha \int_1^A C_1 D_1 \cos \theta \, ds + \beta \int_1^A X D_1 \sin \theta \, ds \right] \\
 & + W_2 \left[ \alpha \int_2^A C_2 D_2 \cos \theta \, ds + \beta \int_2^A X D_2 \sin \theta \, ds \right] \\
 & + \dots \dots \dots \\
 & + W_n \left[ \alpha \int_n^A C_n D_n \cos \theta \, ds + \beta \int_n^A X D_n \sin \theta \, ds \right],
 \end{aligned}$$

where  $C_1 D_1, \dots C_n D_n, X D_1, \dots X D_n$ , represent the lengths corresponding to  $CD$ ,  $XD$  drawn for the points 1, 2  $\dots n$ .

The terms in  $W$  in the other two equations are expanded in a similar manner.

The various integrals in the foregoing equations cannot in general be evaluated directly, and recourse must be had to graphical methods, as shown in Part II, Example 5.

The integration must be extended round the whole curve for the terms involving  $M_0$ ,  $T_0$ , and  $F_0$ , but only round the appropriate ranges for the terms in  $W_1$ ,  $W_2$ , etc. The values of the integrals thus found are substituted in equations (17), (18), and (19), which are then solved for  $M_0$ ,  $T_0$ , and  $F_0$ .

The resultant actions at any point in the girder can then be found from the equations for  $M_x$  and  $T_x$ , and by algebraically summing external forces to obtain shear.

**Determination of the Deflection at a Load Point on Bow Girder.**

—The deflection of the girder at the point  $C$  at which a load  $W$  is acting is given by

$$\Delta_c = \frac{\partial U}{\partial W}.$$

Now

$$\frac{\partial U}{\partial W} = \alpha \int M_x \frac{\partial M_x}{\partial W} ds + \beta \int T_x \frac{\partial T_x}{\partial W} ds$$

and between  $B$  and  $C$

$$\frac{\partial M_x}{\partial W} = 0, \quad \frac{\partial T_x}{\partial W} = 0,$$

while between  $C$  and  $A$

$$\frac{\partial M_x}{\partial W} = CD, \quad \frac{\partial T_x}{\partial W} = -XD.$$

Substituting these values in the above expression for  $\frac{\partial U}{\partial W}$  and using the equations for  $M_x$  and  $T_x$ , we obtain

$$\begin{aligned} \Delta_c = & M_0 \left[ \alpha \int_C^A CD \cos \theta \, ds + \beta \int_C^A XD \sin \theta \, ds \right] \\ & + T_0 \left[ \alpha \int_C^A CD \sin \theta \, ds - \beta \int_C^A XD \cos \theta \, ds \right] \\ & - F_0 \left[ \alpha \int_C^A BE \cdot CD \, ds + \beta \int_C^A XE \cdot XD \, ds \right] \\ & + W \left[ \alpha \int_C^A CD^2 \, ds + \beta \int_C^A XD^2 \, ds \right], \end{aligned}$$

whence, by substituting the values of  $M_0$ ,  $T_0$ , and  $F_0$  already found



expressions for  $M_x$  and  $T_x$ , *e.g.* the integral involving  $W_1$  is only carried over the range  $C_1A$ .

Hence by direct comparison with the equation for  $\Delta_0$  we obtain

$$\begin{aligned}\Delta_{C_2} = & M_0 \left[ \alpha \int_A^{C_2} C_2 D_2 \cos \theta \, ds + \beta \int_A^{C_2} X D_2 \sin \theta \, ds \right] \\ & + T_0 \left[ \alpha \int_A^{C_2} C_2 D_2 \sin \theta \, ds - \beta \int_A^{C_3} X D_2 \cos \theta \, ds \right] \\ & - F_0 \left[ \alpha \int_A^{C_2} B E \cdot C_2 D_2 \, ds + \beta \int_A^{C_2} X E \cdot X D_2 \, ds \right] \\ & + W_3 \left[ \alpha \int_A^{C_2} C_3 D_3 \cdot C_2 D_2 \, ds + \beta \int_A^{C_2} X D_3 \cdot X D_2 \, ds \right] \\ & + W_2 \left[ \alpha \int_A^{C_2} C_2 D_2^2 \, ds + \beta \int_A^{C_2} X D_2^2 \, ds \right] \\ & + W_1 \left[ \alpha \int_A^{C_1} C_1 D_1 \cdot C_2 D_2 \cdot ds + \beta \int_A^{C_1} X D_1 \cdot X D_2 \, ds \right] \quad (20)\end{aligned}$$

and similarly for  $\Delta_{C_1}$  and  $\Delta_{C_3}$ .

#### **Bow Girder with Concentrated Loads and Intermediate Supports.**

—In many cases a bow girder is supported at intermediate points in its span either by columns or by cantilevers from the wall. In such cases the magnitudes of the reactions at these supports must be calculated.

The supports may give restraint of two kinds—

(a) The column may be so attached to the girder that no movement of any sort can take place, *i.e.* the support cannot sink or the slope of the girder be altered by an external load system. This type of restraint would occur in a monolithic construction, *e.g.* in a reinforced concrete design.

(b) The column may be in effect pin jointed to the girder, so that whilst the column exerts a vertical reaction the girder is allowed complete freedom to adjust its slope. In this case the support may be rigid or elastic, *i.e.* under its load it may either remain unaltered in length or it may deflect slightly. The former case would be represented by a heavy column and the latter by cantilever supports.

The conditions in any actual case will probably be neither exactly those of (a) or (b), but the inherent uncertainties of any intermediate state prohibit exact analysis. It should generally be possible to decide which of the two kinds of support is nearer the actual.

The case of (a) can be dealt with simply. Supports of the

kind stated are equivalent to the wall fixing previously dealt with and exactly the same methods are used ; each section of the girder between supports can be treated as a separate unit and dealt with independently.

When the reactions exert a vertical constraint only, we can proceed as follows : each restraint, in this case each support, introduces one more unknown to be determined ; hence we need one more equation for each reaction.

In Fig. 26 let there be two supports at  $C_1$  and  $C_3$  respectively and a load  $W_2$  at  $C_2$ .

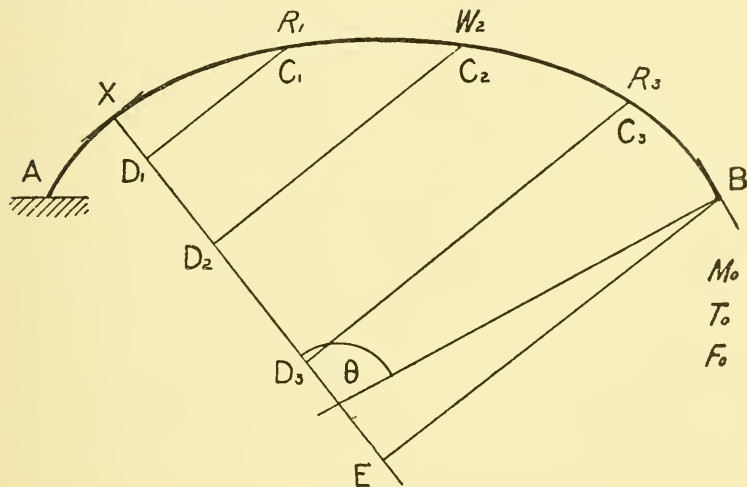


FIG. 26.

Let the magnitudes of the reactions at  $C_1$  and  $C_3$  be  $R_1$  and  $R_3$ . Then treating  $R_1$  and  $R_3$  as forces on the girder acting in the opposite direction to  $W_2$  we can obtain expressions for the deflection of  $C_1$  and  $C_3$  in terms of  $M_0$ ,  $T_0$ ,  $F_0$ ,  $W_2$ ,  $R_1$ , and  $R_3$ .

Thus

$$\frac{\partial U}{\partial R_1} = \text{movement of } C_1 \text{ in the direction of } R_1,$$

$$\frac{\partial U}{\partial R_3} = \text{movement of } C_3 \text{ in the direction of } R_3.$$

If the supports are inelastic there is no movement, and

$$\frac{\partial U}{\partial R_1} = 0,$$

$$\frac{\partial U}{\partial R_3} = 0.$$

If the supports sink, the movement of  $C_1$  and  $C_3$  are in the opposite direction to the loads  $R_1$  and  $R_3$ , and supposing such elastic movements to be  $\delta_1$  and  $\delta_3$ , we have

$$\frac{\partial U}{\partial R_1} = -\delta_1,$$

$$\frac{\partial U}{\partial R_3} = -\delta_3,$$

where  $\delta_1$  and  $\delta_3$  are known, or calculable in terms of  $R_1$  and  $R_3$ .

If we imagine inelastic supports, we can write down our equations as follows :

The first three are as in (17), (18), and (19) with the  $W$  term expanded to contain expressions for  $R_1$ ,  $R_3$ , and  $W_2$ : it must be remembered that  $R_1$  and  $R_3$  are negative. The remaining two equations are of the form (20),  $R_1$  and  $R_3$  again being negative.

We thus obtain

$$\begin{aligned} & M_0(A_{17}) + T_0(B_{17}) - F_0(C_{17}) \\ & + W_2 \left[ \alpha \int_A^{C_2} C_2 D_2 \cos \theta \, ds + \beta \int_A^{C_2} X D \sin \theta \, ds \right] \\ & - R_1 \left[ \alpha \int_A^{C_1} C_1 D_1 \cos \theta \, ds + \beta \int_A^{C_1} X D_1 \sin \theta \, ds \right] \\ & - R_3 \left[ \alpha \int_A^{C_3} C_3 D_3 \cos \theta \, ds + \beta \int_A^{C_3} X D_3 \sin \theta \, ds \right] = 0. \end{aligned}$$

$$\begin{aligned} & M_0(A_{18}) + T_0(B_{18}) - F_0(C_{18}) \\ & + W_2 \left[ \alpha \int_A^{C_2} C_2 D_2 \sin \theta \, ds - \beta \int_A^{C_2} X D_2 \cos \theta \, ds \right] \\ & - R_1 \left[ \alpha \int_A^{C_1} C_1 D_1 \sin \theta \, ds - \beta \int_A^{C_1} X D_1 \cos \theta \, ds \right] \\ & - R_3 \left[ \alpha \int_A^{C_3} C_3 D_3 \sin \theta \, ds - \beta \int_A^{C_3} X D_3 \cos \theta \, ds \right] = 0. \end{aligned}$$

$$\begin{aligned} & -M_0(A_{19}) - T_0(B_{19}) + F_0(C_{19}) \\ & - W_2 \left[ \alpha \int_A^{C_2} C_2 D_2 \cdot B E ds + \beta \int_A^{C_2} X D_2 \cdot X E ds \right] \\ & + R_1 \left[ \alpha \int_A^{C_1} C_1 D_1 \cdot B E ds + \beta \int_A^{C_1} X D_1 \cdot X E ds \right] \\ & + R_3 \left[ \alpha \int_A^{C_3} C_3 D_3 \cdot B E ds + \beta \int_A^{C_3} X D_3 \cdot X E ds \right] = 0. \end{aligned}$$

$$\begin{aligned}
 & M_0(A_{20}) + T_0(B_{20}) - F_0(C_{20}) \\
 & + W_2 \left[ \alpha \int_A^{C_1} C_2 D_2 \cdot C_1 D_1 ds + \beta \int_A^{C_1} X D_2 \cdot X D_1 ds \right] \\
 & - R_1 \left[ \alpha \int_A^{C_1} C_1 D_1^2 ds + \beta \int_A^{C_1} X D_1^2 ds \right] \\
 & - R_3 \left[ \alpha \int_A^{C_1} C_3 D_3 \cdot C_1 D_1 ds + \beta \int_A^{C_1} X D_3 \cdot X D_1 ds \right] = 0.
 \end{aligned}$$

$$\begin{aligned}
 & M_0(A'_{20}) + T_0(B'_{20}) - F_0(C'_{20}) \\
 & + W_2 \left[ \alpha \int_A^{C_2} C_2 D_2 \cdot C_3 D_3 ds + \beta \int_A^{C_2} X D_2 \cdot X D_3 ds \right] \\
 & - R_1 \left[ \alpha \int_A^{C_1} C_1 D_1 \cdot C_3 D_3 ds + \beta \int_A^{C_1} X D_1 \cdot X D_3 ds \right] \\
 & - R_3 \left[ \alpha \int_A^{C_1} C_3 D_3^2 ds + \beta \int_A^{C_3} X D_3^2 ds \right] = 0
 \end{aligned}$$

where

$A_{17}, B_{17}, C_{17}$  denote the coefficients of  $M_0, T_0$ , and  $F_0$  from (17)

$A_{18}, B_{18}, C_{18}$  denote the coefficients of  $M_0, T_0$ , and  $F_0$  from (18)

$A_{19}, B_{19}, C_{19}$  denote the coefficients of  $M_0, T_0$ , and  $F_0$  from (19)

$A_{20}, B_{20}, C_{20}$  denote the coefficients of  $M_0, T_0$ , and  $F_0$  from (20)

with  $C_1 D_1$  and  $X D_1$  replacing  $C_2 D_2$  and  $X D_2$ .

$A'_{20}, B'_{20}, C'_{20}$  denote the coefficients of  $M_0, T_0$ , and  $F_0$  from (20)

with  $C_3 D_3$  and  $X D_3$  replacing  $C_2 D_2$  and  $X D_2$ .

The five equations can now be solved to obtain  $M_0, T_0, F_0, R_1$  and  $R_2$ , and the stress at any point in the girder is obtained as before.

**Bow Girder with a Uniformly Distributed Load.**—So far we have been concerned only with a load system consisting of concentrated weights, but a more usual case is that of a uniformly distributed load. Let Fig. 27 represent a girder carrying such a load of intensity  $w$ .

Then as before we cut the girder normally at B and replace the fixing effects by  $M_0, T_0$ , and  $F_0$ .

The terms in  $M_0, T_0$ , and  $F_0$  are as in equations (17), (18), and (19), but those relating to the external load are modified.

Consider the bending moment and torque at any section X due to the distributed load.

Let  $m_x$  = bending moment at X due to distributed load,

$-t_x$  = torque at X                      ,,                      ,,                      ,,

Let  $ds$  be a small element of the curve at C.

Then 
$$m_x = \int_B^X wCDds$$

and 
$$-t_x = \int_B^X wXDds.$$

These integrals must be evaluated graphically by plotting values of CD and XD against  $s$  and measuring the areas of the resulting curves. These multiplied by  $w$  then give the bending

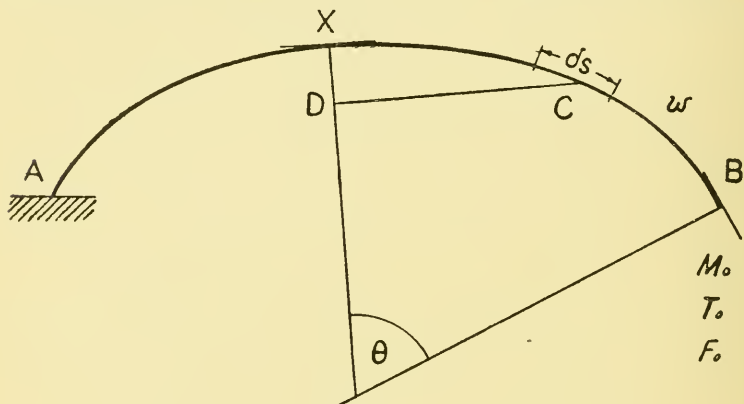


FIG. 27.

moment and torque due to the distributed load acting alone, and must be found for a number of different points around the girder.

Then we can rewrite the equations for  $M_x$  and  $T_x$  in the form

$$M_x = M_0 \cos \theta + T_0 \sin \theta - F_0 BE + m_x$$

$$T_x = -M_0 \sin \theta + T_0 \cos \theta + F_0 XE - t_x$$

and instead of equations (17), (18), and (19) we obtain

$$M_0(A_{17}) + T_0(B_{17}) - F_0(C_{17}) + \alpha \int m_x \cos \theta ds + \beta \int t_x \sin \theta ds = 0,$$

$$M_0(A_{18}) + T_0(B_{18}) - F_0(C_{18}) + \alpha \int m_x \sin \theta ds - \beta \int t_x \cos \theta ds = 0,$$

$$-M_0(A_{19}) - T_0(B_{19}) + F_0(C_{19}) - \alpha \int m_x BE ds - \beta \int t_x XE ds = 0,$$

from which  $M_0$ ,  $T_0$ , and  $F_0$  can be determined after the various integrals have been found graphically.

The integrals in the above equations extend round the whole girder. If the loading is not uniform in intensity or extends only

over a portion of the girder, the necessary corrections should be made in the evaluation of  $m_x$  and  $t_x$ . If a girder loaded with a distributed load is provided with intermediate supports, the method of solution follows directly from what has previously been said upon this matter.

**Special Cases.**—So far the treatment has been quite general, but very frequently conditions are such that a great reduction in labour is possible. Suppose we have a girder as shown in Fig. 28 which is symmetrical about CD, the loading also being symmetrical about this line. Then it is clear that the reactions  $F_A$  and  $F_B$

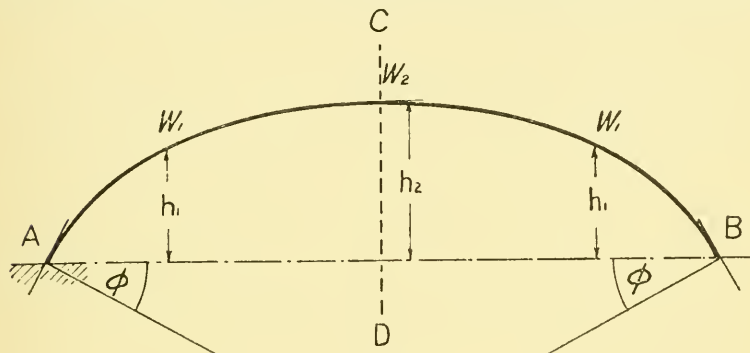


FIG. 28.

are equal, as well as the fixing moments  $M_A$  and  $M_B$  and the torques  $T_A$  and  $T_B$ .

Hence, since  $F_A + F_B = 2W_1 + W_2$ ,

we have  $F_A = F_B = \frac{1}{2}(2W_1 + W_2)$ .

Again, taking moments about the line AB,

$$(M_A + M_B) \cos \phi = 2W_1 h_1 + W_2 h_2 + (T_A + T_B) \sin \phi,$$

therefore  $M_A = M_B = \frac{1}{2}(2W_1 h_1 + W_2 h_2) \sec \phi + T_B \tan \phi$ ,

where  $\phi$  is the angle between AB and the normal at A or B. Now  $F_B$ ,  $M_B$ , and  $T_B$  are the actions we have previously called  $F_0$ ,  $M_0$ , and  $T_0$ ; hence the only statically indeterminate term in this case is  $T_0$ , and we resort to the previous conditions of fixity at the ends in order to find its value.

Our equations (17) and (19) which state the conditions that

$$\frac{\partial U}{\partial M_0} = 0 \quad \text{and} \quad \frac{\partial U}{\partial F_0} = 0$$

are now not required, and we substitute the values of  $M_0$  and  $F_0$  in (18) and evaluate  $T_0$ . This procedure can be followed

for any symmetrical lay-out, including symmetrically spaced reactions, and the work thereby considerably lightened.

**Determination of Torsional Rigidity.**—The determination of the value of the torsional rigidity in general presents some difficulty, since  $J$  is not the polar moment of inertia of the section except in the special case of a circle. Gibson and Ritchie, in the work previously quoted, give a number of tests from which experimental values of  $J$  have been determined for various sections, and reference may be made to these for the desired values. In the event of a special section being used which is not included in this series, the values of  $J$  can be found by a semi-graphical method <sup>1</sup> or by a special test.

**Curved Bars used as Struts or Ties.**—In some cases, *e.g.* in timber roofs of the open-framed type, curved members are used instead of straight with the object of obtaining pleasing effects, or for other reasons.

Such members require special treatment.<sup>2</sup>

If a curved member, as in Fig. 29, transmits a load between two points A and B, it is clear that the stress in the material is composed of two parts: a direct compressive stress and a bending stress.

If  $a$  is the maximum distance of the line joining A and B from the axis of the member, the maximum bending moment is  $Pa$  and the maximum stress is, if the curvature is not large, approximately

$$f = \left( \frac{P}{A} + \frac{Pay}{I} \right)$$

where  $I$  is the moment of inertia of the member about the axis of bending,  $y$  is the distance of its most stressed fibre from the neutral axis, and  $A$  is its cross-sectional area.

When the curvature is large the formulæ for stresses in curved bars should be used.

If, then, we know  $P$ , we can calculate  $f$ . Suppose now that the curved member is a bar in a truss which is redundantly braced;  $P$  can only be found by an application of the method just outlined. If the member connecting A and B were straight, the method would be straightforward, but in a curved member the amount

<sup>1</sup> "The Determination of Torsional Stresses in a Shaft of any Cross-section." Bairstow and Pippard. Vol. CCXIV, *Proc. Inst. C.E.*, 1921-22.

<sup>2</sup> *Primary Stresses in Timber Roofs.* A. J. S. Pippard and W. H. Glanville. H.M. Stationery Office, 1926.

of compressive strain under a load  $P$  is greater than it would be if the member were of the same cross-section but straight.

In other words, the shortening of the initial distance between  $A$  and  $B$  will be more if they are connected by a curved member than if the member were the same size but straight. This is equivalent to the effect which would be obtained on replacing the curved bar by a straight bar of the same size but of a lower modulus of elasticity. The method for treating such members is, then, as follows: *replace the curved members by straight members of the same dimensions but of a lower value of  $E$ ; calculate the loads in such straight members by the usual methods, and then determine the stress in the material by the use of the above equation.*

It is necessary, therefore, in the first place to find the equivalent

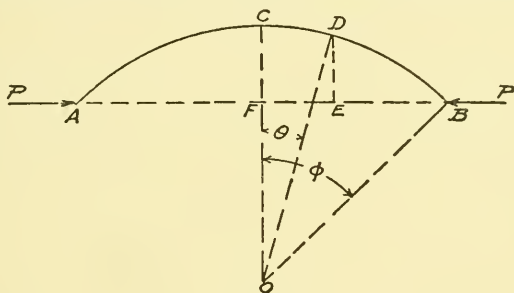


FIG. 30.

value of the modulus of elasticity of a curved member. In Fig. 30 let  $ACB$  be the centre line of a circular arc strut subtending an angle  $2\phi$  at the centre of curvature  $O$ .

Let  $C$  be the middle point of the arc so that  $COB = \phi$  and let the radius of curvature be  $R$ .

Suppose this strut to carry a load  $P$  as shown.

Let  $D$  be any point on the arc where  $COD = \theta$ .

At  $D$  the axial compression  $= P \cos \theta$

and the bending moment  $= P \cdot DE$

$$= PR (\cos \theta - \cos \phi).$$

The strain energy of a small length of the arc  $ds$  at  $D$  is

$$\frac{(P \cos \theta)^2 ds}{2AE} + \frac{[PR(\cos \theta - \cos \phi)]^2 ds}{2EI}$$

The first term in the above expression is the strain energy due to compression, and the second term is the strain energy due to bending. The small term due to shear is neglected.

Since  $ds = R d\theta$ , the total strain energy of the curved strut is

$$U = \frac{R}{AE} \int_0^\phi P^2 \cos^2 \theta \, d\theta + \frac{R^3}{EI} \int_0^\phi P^2 (\cos \theta - \cos \phi)^2 d\theta.$$

Hence the amount by which A and B approach each other under the action of the force P is

$$e = \frac{dU}{dP} = \frac{2PR}{AE} \int_0^\phi \cos^2 \theta \, d\theta + \frac{2PR^3}{EI} \int_0^\phi (\cos \theta - \cos \phi)^2 d\theta,$$

which upon integration and reduction gives the expression

$$e = \frac{PR}{E} \left[ \frac{1}{A} \left( \phi + \frac{1}{2} \sin 2\phi \right) + \frac{R^2}{I} \left( 2\phi - \frac{3}{2} \sin 2\phi + \phi \cos 2\phi \right) \right].$$

Now imagine that the curved member ACB is replaced by a straight member of the same cross-sectional area and having a modulus of elasticity  $E'$  such that its compression under the force P will be equal to  $e$ .

Since the length of this hypothetical member is  $2R \sin \phi$  its compression under a load P is

$$\frac{2PR \sin \phi}{AE'}.$$

Equating this to the value of  $e$  given above, we have

$$\frac{2PR \sin \phi}{AE'} = \frac{PR}{E} \left[ \frac{1}{A} \left( \phi + \frac{1}{2} \sin 2\phi \right) + \frac{R^2}{I} \left( 2\phi - \frac{3}{2} \sin 2\phi + \phi \cos 2\phi \right) \right],$$

and if  $k$  is the radius of gyration about the axis of bending this becomes

$$\frac{E'}{E} = \frac{2 \sin \phi}{\left( \phi + \frac{1}{2} \sin 2\phi \right) + \frac{R^2}{k^2} \left( 2\phi - \frac{3}{2} \sin 2\phi + \phi \cos 2\phi \right)}, \quad (21)$$

giving a value for  $E'$  the modulus of elasticity of a straight member of the same section as the curved one, which would behave elastically in the same way as the circular arc strut.

In order to verify the accuracy of the formula of equation (21), two struts were made of mild steel and carefully tested. The struts are shown in Fig. 31. (a) was a semicircle and (b) was a segment subtending an angle of  $120^\circ$  at the centre. The internal radius of each was 4.984 in. and the dimensions were 0.998 in.  $\times$  0.727 in., the cross-section being rectangular.

These experimental struts were subjected to compressive

forces in a testing machine, and the amount of the compressive strain measured for a number of loads. The points thus obtained

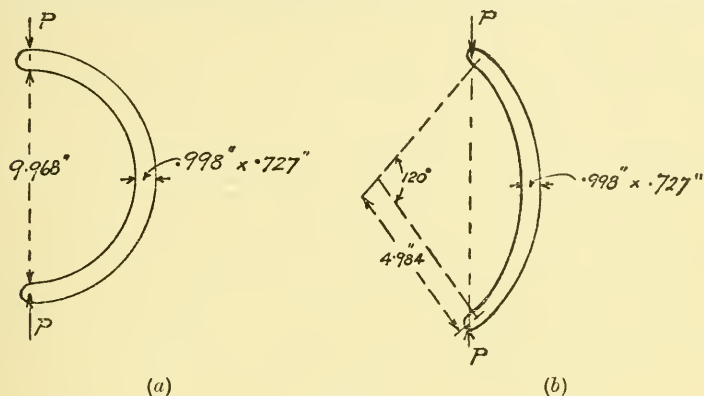


FIG. 31.

show a linear relation between load and strain for each strut (Figs. 32 and 33).

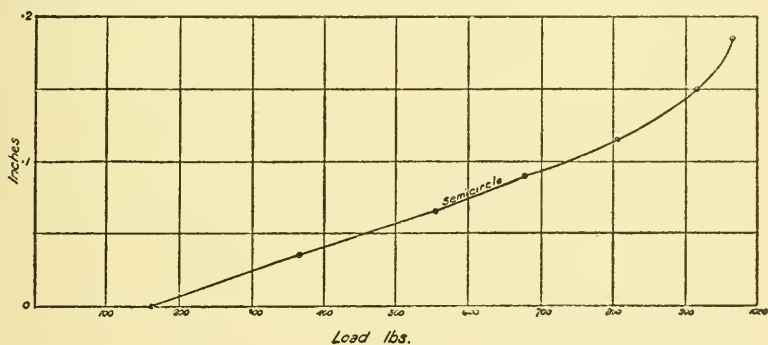


FIG. 32.

The experimental value of  $E'$  was then found from the expression  $E' = \frac{\text{stress}}{\text{strain}}$ ,

where  $\text{stress} = \frac{\text{load}}{\text{area}}$ , and  $\text{strain} = \frac{\text{measured decrease under load}}{2R \sin \phi}$ .

The experimental values of  $E'$  thus obtained were :

for the semicircular arc  $E' = 93,400$  lbs. per sq. in.,  
for the  $120^\circ$  arc  $E' = 441,400$  lbs. per sq. in.

A piece of the same material from which the two struts were

cut was then made into a standard tension test specimen, and the modulus of elasticity for the material was found to be 26,960,000 lbs. per sq. in.

This is the value  $E$  in the equation (21) and so the experimental values of the ratio  $\frac{E'}{E}$  are :

$$\text{for the semicircle} \quad \frac{E'}{E} = 0.00346,$$

$$\text{for the } 120^\circ \text{ arc} \quad \frac{E'}{E} = 0.0164.$$

By calculation from equation (21) we obtain :

$$\text{for the semicircle} \quad \frac{E'}{E} = 0.0035,$$

$$\text{for the } 120^\circ \text{ arc} \quad \frac{E'}{E} = 0.0173.$$

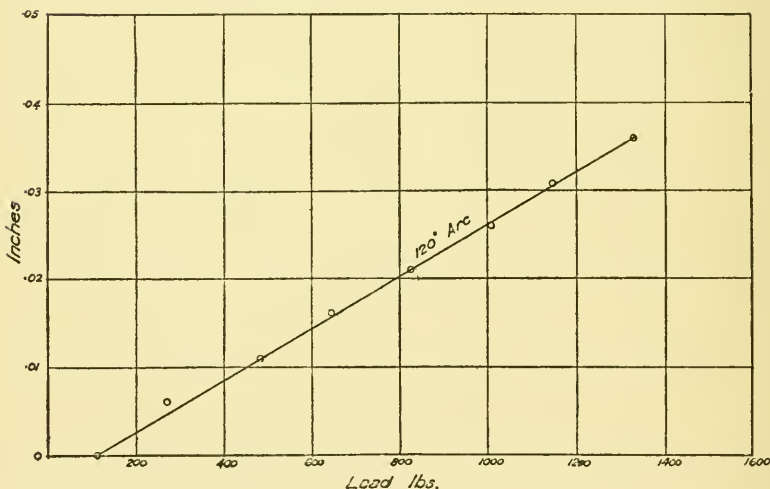


FIG. 33.

These show errors between the theoretical and experimental values of 1.1 per cent. and 5.5 per cent. respectively for the semicircle and  $120^\circ$  arc.

If a term for the shear strain energy is introduced into equation (21), these errors are reduced to 0.5 per cent. and 4.3 per cent.

The semicircle thus gives practical agreement, and the error of 4.3 per cent. in the flatter arc was due to difficulties of measurement in the test.

In the foregoing analysis the strut was assumed to be a circular arc, but the method has been also applied to arcs of a parabolic form, and it was found that the equivalent modulus of elasticity  $E'$  was not modified seriously if the ratio  $\frac{CF}{AB}$  in Fig. 30 was kept the same in both the circular and parabolic struts.

Equation (21) thus allows us to replace all the curved members in a framework by straight members of the same cross-section but with modified values of  $E$ . The strain energy calculations are then carried out in the usual way, and the loads in the fictitious straight members obtained. The curved members are then replaced and the maximum stresses calculated. If the structure is just-stiff, the question of the elastic properties of the members does not arise, and a stress diagram may be prepared, the actual stress in the material being calculated as previously shown.

Should the deflections of any structure containing curved bars be required, whether the structure be just-stiff or redundant, the curved bars can be replaced by equivalent straight members and the calculation made by standard methods.

For practical use an approximation to equation (21) can be made as follows:—

Inverting the formula we obtain

$$\frac{E}{E'} = \frac{(\phi + \frac{1}{2} \sin 2\phi) + \left(\frac{R}{k}\right)^2 (2\phi - \frac{3}{2} \sin 2\phi + \phi \cos 2\phi)}{2 \sin \phi},$$

which may be written

$$\frac{E}{E'} = M + N \left(\frac{R}{k}\right)^2$$

$$\text{where } M = \left(\frac{\phi + \frac{1}{2} \sin 2\phi}{2 \sin \phi}\right) \text{ and } N = \left[\frac{2\phi - \frac{3}{2} \sin 2\phi + \phi \cos 2\phi}{2 \sin \phi}\right],$$

$M$  being due to compression and  $N \left(\frac{R}{k}\right)^2$  due to bending.

A number of curved members in roofs have been examined and none found in which  $\phi$  is greater than  $45^\circ$  or less than  $9.5^\circ$ , while the corresponding extreme values of  $\frac{E'}{E}$  are 213.5 and 1.8.

Values of  $M$  and  $N$  have been worked out between these extreme values of  $\phi$  and are plotted in Fig. 34. It will be seen that  $M$  varies between 1.0 and 0.9, a comparatively small and



FIG. 34.

unimportant variation, and since it is an additive constant, the formula may be simplified with but small loss in accuracy by writing

$$\frac{E}{E'} = 1 + N \left( \frac{R}{k} \right)^2 \quad \dots \quad (22)$$

$M$  is taken as unity, since it has practically this value when  $\frac{E}{E'}$  is nearly 1 and the influence of  $M$  is greatest.

It is only necessary, therefore, in determining the equivalent modulus of elasticity, to take  $\phi$  from the drawing, read off the corresponding value of  $N$  from the curve, and substitute, together with  $R$  and  $k$ , in the formula.

**Stresses in a Circular Ring.**—The stresses in a circular ring loaded in any manner can readily be found by strain energy methods. As an example take the case shown in Fig. 35.

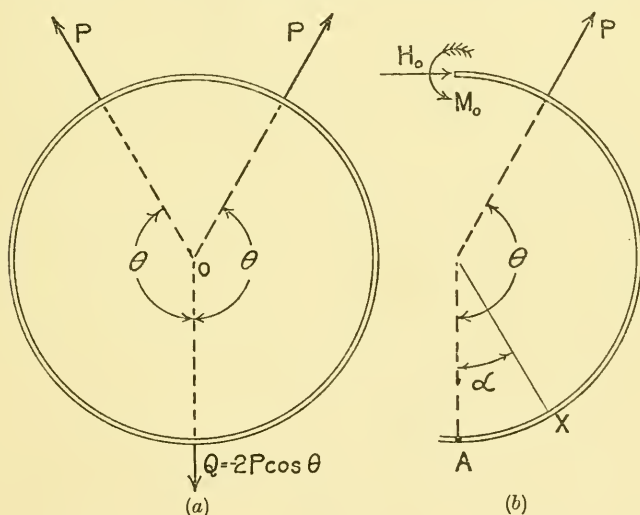


FIG. 35.

Two loads of equal amount  $P$  are applied to a ring of radius  $R$  so that their lines of action pass through the centre of the ring and make angles  $\theta$  with the vertical equilibrating load  $Q$ .

If the link is cut at the top section as shown at Fig. 35 (b) it is necessary to apply certain actions to the cut section to restore the conditions of equilibrium that obtained before cutting. These actions are a couple  $M_0$  and a horizontal force  $H_0$ . Considerations of symmetry show that there is no vertical action.

Consider the half ring under the action of a force  $P$ , and the reactions  $M_0$  and  $H_0$ , and suppose this half ring to be held at  $A$ .

If  $U$  is the total strain energy of the half ring, since the symmetry precludes either angular or horizontal movement of the middle section, we have

$$\frac{\partial U}{\partial M_0} = \text{angular rotation of cut section} = 0.$$

$$\frac{\partial U}{\partial H_0} = \text{horizontal movement of cut section} = 0.$$

The direct load in the ring and the shear will have small effect upon the strain energy compared with the bending, and so the last only will be taken into account.

Then 
$$U = \frac{1}{2EI} \int M^2 ds,$$

and so 
$$\frac{\partial U}{\partial M_0} = \frac{1}{EI} \int M \frac{\partial M}{\partial M_0} ds$$

and 
$$\frac{\partial U}{\partial H_0} = \frac{1}{EI} \int M \frac{\partial M}{\partial H_0} ds.$$

Let  $\angle AOX = \alpha$ .

Then between  $\alpha = 0$  and  $\alpha = \theta$

$$M_x = M_0 - H_0 R(1 + \cos \alpha) - PR \sin(\theta - \alpha)$$

while between  $\alpha = \theta$  and  $\alpha = \pi$

$$M_x = M_0 - H_0 R(1 + \cos \alpha).$$

Also 
$$\frac{\partial M_x}{\partial M_0} = 1 \text{ throughout}$$

and 
$$\frac{\partial M_x}{\partial H_0} = -R(1 + \cos \alpha).$$

Since  $ds = R d\alpha$ , we have

$$\begin{aligned} \frac{\partial U}{\partial M_0} &= \frac{R}{EI} \int_0^\theta \{M_0 - H_0 R(1 + \cos \alpha) - PR \sin(\theta - \alpha)\} d\alpha \\ &\quad + \frac{R}{EI} \int_\theta^\pi \{M_0 - H_0 R(1 + \cos \alpha)\} d\alpha \\ &= \frac{R}{EI} \left[ \int_0^\pi \{M_0 - H_0 R(1 + \cos \alpha)\} d\alpha - PR \int_0^\theta \sin(\theta - \alpha) d\alpha \right]. \end{aligned}$$

Upon integration and equation to zero this gives

$$\pi M_0 - \pi H_0 R - PR(1 - \cos \theta) = 0.$$

Again,

$$\frac{\partial U}{\partial H_0} = -\frac{R^2}{EI} \int_0^\theta \{M_0 - H_0 R(1 + \cos \alpha) - PR \sin(\theta - \alpha)\}(1 + \cos \alpha) d\alpha \\ - \frac{R^2}{EI} \int_\theta^\pi \{M_0 - H_0 R(1 + \cos \alpha)\}(1 + \cos \alpha) d\alpha,$$

which upon integration gives

$$\pi M_0 - \frac{3\pi H_0 R}{2} - PR \left(1 - \cos \theta + \frac{\theta \sin \theta}{2}\right) = 0.$$

The solution of these two equations gives

$$H_0 = -\frac{P}{\pi} (\theta \sin \theta),$$

$$M_0 = \frac{PR}{\pi} \{1 - \cos \theta - \theta \sin \theta\}.$$

If in these equations we put  $\theta = \pi$ , we obtain the case of a ring used as a link and subjected to a diametral pull  $2P$ ; then  $H_0$  becomes zero and  $M_0 = \frac{2PR}{\pi}$ .

After the values of  $H_0$  and  $M_0$  have been found, the bending moment at any point in the ring can be obtained, and thence the stresses in the material.

In calculating the stresses, however, it must be remembered that the formulæ applicable to curved bars must be used and not those for straight beams. Failure to use the former may introduce serious errors.

Any type of symmetrical loading on a ring can be dealt with in the way just described, but if there is no symmetry the treatment

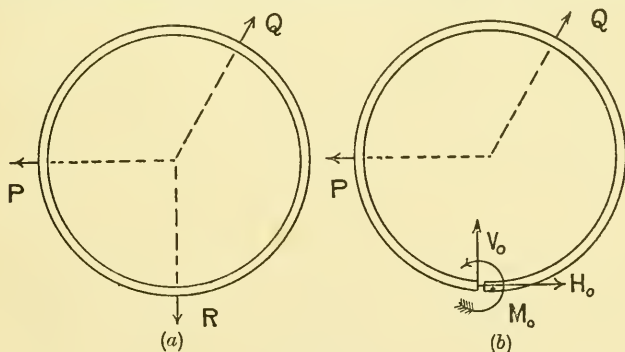


FIG. 36.

is slightly different. For example, suppose the ring to be loaded as shown in Fig. 36, *i.e.* with three radial loads  $P$ ,  $Q$ , and  $R$ , which

are in equilibrium, but are not so arranged as to give an axis of symmetry. We can now imagine the ring cut to the left of the point of application of  $R$  as shown and apply the radial force  $V_0$ , the horizontal force  $H_0$ , and the moment  $M_0$ , which represent the internal reactions at this point. These reactions are redundancies, since, if the ring were actually cut, the problem would be statically determinate. Hence we can write

$$\frac{\partial U}{\partial H_0} = \frac{\partial U}{\partial V_0} = \frac{\partial U}{\partial M_0} = 0,$$

and these equations provide the necessary number to solve for  $H_0$ ,  $V_0$ , and  $M_0$ . The method is precisely as for the previous case, but now it is necessary to carry the integration right round the ring.

An exactly similar method is adopted for the case of links which are not circular, but in many such instances the integration cannot be carried out except by graphical methods.

The problem of the stresses and distortion of links is dealt with in Part II, Example 3, where several cases are worked out in detail.

**Redundancy of Stiff-jointed Frames.**—A pin joint is able to transmit an axial and also a transverse or shearing load, but not a bending moment. If, therefore, any joint in a frame is made stiff instead of pinned, the effect is to introduce one more reaction at that point, viz. the bending moment. Thus, if we have a

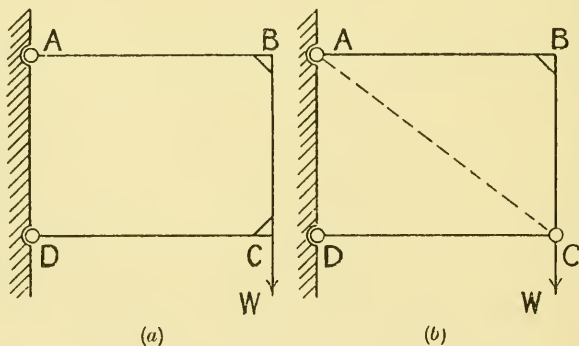


FIG. 37.

frame consisting of the correct number of bars on the assumption that all the joints are pinned, the introduction of every stiff joint gives one redundancy. A stiff joint then may be treated, for the purpose of calculating the number of members, as one bar. As an example consider the case shown in Fig. 37, where

a quadrilateral frame ABCD is supported on pins at A and D. The joints B and C are stiff. If B and C were pinned, the frame would be incomplete by one member, since four bars would be needed to connect the two points B and C to A and D. There are, however, two stiff joints, so that the equivalent number of bars is five, *i.e.* it is redundant by one member.

This will perhaps be more clearly realized by an examination of the case shown at (b) in the figure, where C is a pin joint and B is a stiff joint. The member ABC is now a single member, and from the standpoint of bracing might be replaced by a bar AC. It is evident that the point C is properly fixed by CD and AC, or by CD and the cranked bar ABC. The stiffness of joint B has thus provided the one extra member necessary to convert the mechanism ABCD into a just-stiff structure. If

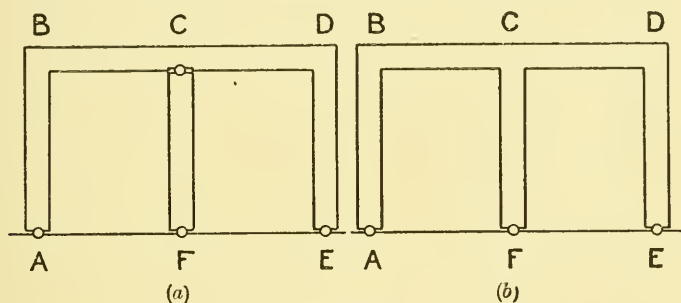


FIG. 38.

in addition we make C stiff, it is equivalent to introducing one redundancy.

Now consider the structure shown in Fig. 38 (a), where A, F, and E are pinned supports while B and D are stiff joints. The member BCD is continuous over C to which it is pinned.

In this case there are three points B, C, and D to be braced to the points A, F, and E, and if the frame is wholly pin-jointed six members are needed. Actually there are five—AB, BC, CD, DE, and CF.

Now consider the stiff joints. An extra member is in effect introduced by each of the joints B and D, and since the top member is continuous at C, this is equivalent to another. Effectively, therefore, there are eight members in the structure and so two redundancies. If, as shown at (b), the member FC, instead of being pinned to the top member BD, is stiff-jointed to it, this introduces yet another redundancy, and so in Fig. 38 (b) there are three redundancies to be considered.

An interesting case of bracing is provided by a spoked wheel of the artillery type as shown in Fig. 39.

In this type of wheel the spokes are firmly attached both to the hub and the rim in a way which produces stiff joints at these points of attachment. Suppose such a wheel to be supported at the hub while a load is applied to the rim in any direction. It is necessary to determine the number of redundancies in the wheel before any attempt can be made to analyse the stresses.

If the whole frame were pin-jointed and there were  $N$  spokes, there would be  $N$  points on the rim to be connected to  $N$  fixed points on the hub, and this would need  $2N$  members. Since there are  $N$  arms and  $N$  rim segments, such a pinned frame would

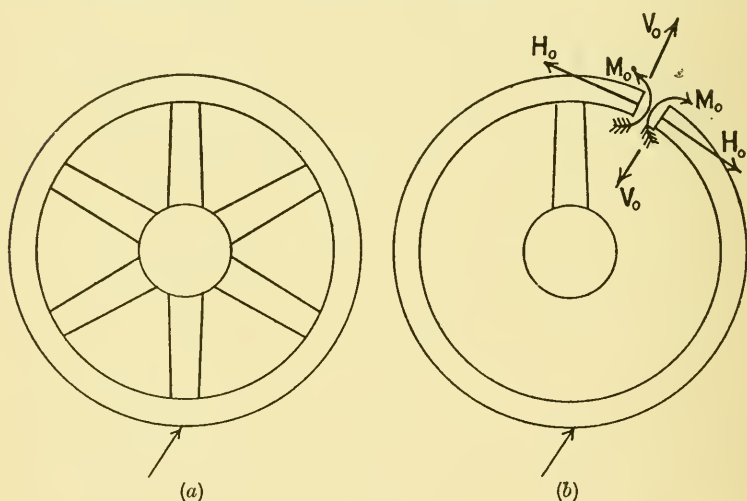


FIG. 39.

satisfy the criterion for the number of essential bars. The rim is, however, continuous over the arms and each continuity introduces one redundancy, so from this cause there are  $N$  redundancies. Again, every arm is stiff-jointed to the rim, which gives another  $N$  redundancies; the stiff joints of the arms to the hub provide  $N$  more, so that in all there are  $3N$  redundancies.

In order to transmit the load from the rim to the hub, then, it would be possible to dispense with  $3N$  reactions, and in Fig. 39 (b) an arrangement is shown which would do this. By cutting out all the spokes except one we have eliminated  $3(N-1)$  redundant reactions, viz. an axial load, a transverse load, and a moment in each spoke. By cutting the rim as shown we eliminate three

more reactions—a bending moment  $M_0$ , a tangential force  $H_0$ , and a radial force  $V_0$ .

In all, then, we have eliminated 3N reactions and the arrangement shown is just-stiff.

**Stress Analysis of Stiff-jointed Frames.**—Suppose the frame shown in Fig. 40 to be pinned to supports at A and B and to have stiff joints at C and D. It carries a load  $W$  at a distance  $l_1$  from C and  $l_2$  from D. The flexural rigidity  $EI$  is the same throughout the frame.

This frame has one redundancy, as we have already seen. The reactions at A and B consist of a vertical and a horizontal

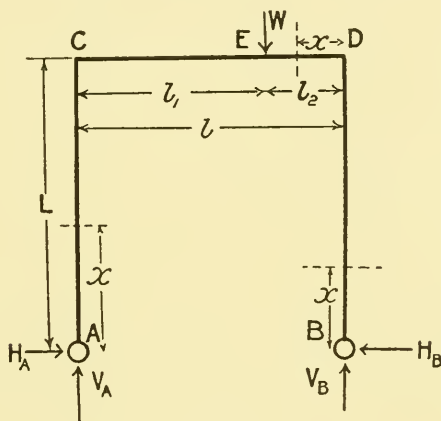


FIG. 40.

force at each point. Call these  $V_A$ ,  $H_A$ , and  $V_B$ ,  $H_B$ . We will consider  $H_B$  as the redundant reaction.

$$\text{Then} \quad \frac{dU}{dH_B} = 0.$$

The bending of the frame is the most important term, so we shall not take into account the direct loads or the shears in the members.

$$\text{Then} \quad \frac{dU}{dH_B} = \frac{1}{EI} M \int \frac{dM}{dH_B} dx,$$

where  $M$  is the bending moment at any point on the frame and the integration extends round the whole frame.

From the conditions of static equilibrium,

$$\begin{aligned} V_A + V_B &= W, \\ H_A &= H_B = H, \text{ say,} \\ V_B l &= W l_1. \end{aligned}$$

Hence 
$$V_B = \frac{l_1}{l} W,$$

$$V_A = \bar{W} \left(1 - \frac{l_1}{l}\right) = \frac{l_2}{l} \bar{W}.$$

Consider the bending in the member BD.

At  $x$  from B, 
$$M_x = H_B x = Hx$$

and 
$$\frac{dM}{dH} = x.$$

So for member BD,

$$\frac{dU}{dH} = \frac{1}{EI} \int_0^L Hx^2 dx = \frac{HL^3}{3EI}.$$

In ED at  $x$  from D,

$$M_x = HL - V_B x = HL - \frac{l_1}{l} \cdot \bar{W} x$$

and 
$$\frac{dM}{dH} = L.$$

So for member ED,

$$\frac{dU}{dH} = \frac{L}{EI} \int_0^{l_2} \left( HL - \frac{l_1}{l} \cdot \bar{W} x \right) dx = \frac{Ll_2}{EI} \left( HL - \frac{\bar{W} l_1 l_2}{2l} \right).$$

If now we measure  $x$  from A, and from C respectively, we shall obtain expressions corresponding to the above as follows :

For AC, 
$$\frac{dU}{dH} = \frac{HL^3}{3EI}.$$

For CE, 
$$\frac{dU}{dH} = \frac{Ll_1}{EI} \left( HL - \frac{Wl_1 l_2}{2l} \right).$$

So for the whole frame,

$$\frac{dU}{dH} = \frac{2HL^3}{3EI} + \frac{Ll}{EI} \left( HL - \frac{\bar{W} l_1 l_2}{2l} \right) = 0,$$

from which 
$$H = \frac{3l_1 l_2}{2L(2L + 3l)} W.$$

Having found  $H$ , the bending moment at any point in the structure can be determined.

If the load had been placed at the mid-point of the top beam as shown in Fig. 41, the treatment could have been simplified slightly as follows.

Since the mid-section at E cannot, from considerations of

symmetry, either move horizontally or rotate, but has only a vertical movement, there is no horizontal movement of point B relative to E. We may thus consider the half structure to be a

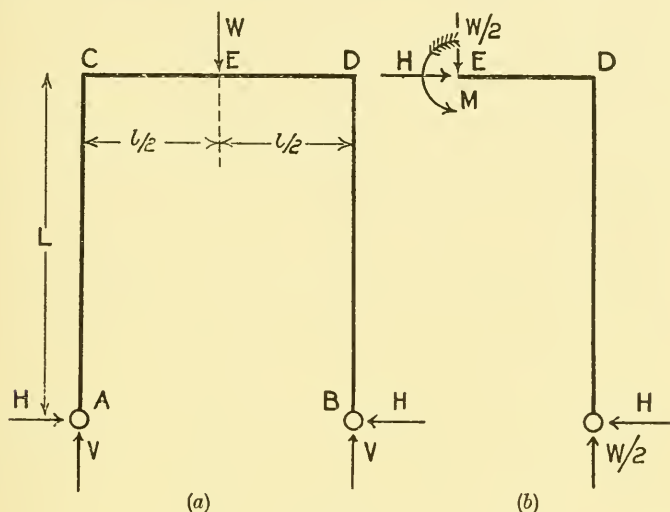


FIG. 41.

cantilever built in at E and loaded at B with a vertical force  $V = \frac{W}{2}$ , and an unknown horizontal force H which can be determined from the condition that

$$\frac{dU}{dH} = 0.$$

Then for BD we get, as in the previous example,

$$\frac{dU}{dH} = \frac{HL^3}{3EI},$$

and for DE,

$$\frac{dU}{dH} = \frac{Ll}{2EI} \left( HL - \frac{Wl}{8} \right);$$

so

$$\frac{dU}{dH} = 0$$

gives

$$\frac{HL^2}{3} + \frac{l}{2} \left( HL - \frac{Wl}{8} \right) = 0,$$

or,

$$H = \frac{3l^2}{8L(2L + 3l)} W.$$

As a further example, consider the structure shown in Fig. 42, where the ends A and D are *encastré* and joints B and C are stiff. The top beam carries a uniformly distributed load of intensity  $w$ .

If all the members were pinned together, four members would be needed to connect the two points B and C to the supports A and D.

The equivalent number of members is seven—the three actual bars AB, BC, and CD, and the four stiff connections at A, B, C, and D. There are thus three redundancies.

In the general case of a structure of this type under any loading the method of attack is as shown at (b) in the figure.

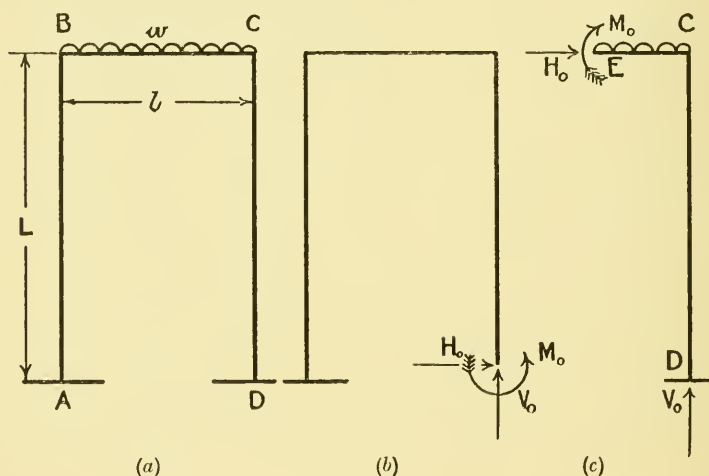


FIG. 42.

The reaction of the support at D can be replaced by a moment  $M_0$ , a vertical force  $V_0$ , and a horizontal force  $H_0$ . The bending moment at any point in the structure is then found in terms of these three reactions, and the conditions to be satisfied by these are,

$$\frac{\partial U}{\partial M_0} = \frac{\partial U}{\partial H_0} = \frac{\partial U}{\partial V_0} = 0.$$

In the particular case shown at (a), however, the work is simplified on account of the symmetry of the structure and loading.  $V_0$  in this case is  $\frac{wl}{2}$  and this eliminates one unknown.

Consider the structure to be cut at the mid-point of the top beam as shown at (c), and let the reacting forces of the left-hand

section on the beam at E be replaced by external resultants  $H_0$  and  $M_0$ . Then from the symmetry of the structure we can write

$$\begin{aligned}\frac{\partial U}{\partial H_0} &= \text{horizontal movement of E} = 0, \\ \frac{\partial U}{\partial M_0} &= \text{angular rotation of cut section} = 0.\end{aligned}$$

As before, considering only the bending energy we can write

$$\begin{aligned}\frac{\partial U}{\partial H_0} &= \frac{1}{EI} \int M_x \frac{\partial M_x}{\partial H_0} dx = 0, \\ \frac{\partial U}{\partial M_0} &= \frac{1}{EI} \int M_x \frac{\partial M_x}{\partial M_0} dx = 0.\end{aligned}$$

For EC, measuring  $x$  from E we have,

$$M_x = M_0 - \frac{wx^2}{2}.$$

Hence 
$$\frac{\partial M_x}{\partial M_0} = 1, \quad \frac{\partial M_x}{\partial H_0} = 0,$$

and 
$$\frac{\partial U}{\partial M_0} = \frac{1}{EI} \int_0^l \left( M_0 - \frac{wx^2}{2} \right) dx = \frac{1}{EI} \left[ \frac{M_0 l}{2} - \frac{wl^3}{48} \right].$$

For CD, measuring  $x$  from C

$$M_x = M_0 - \frac{wl^2}{8} + H_0 x.$$

Hence 
$$\frac{\partial M_x}{\partial M_0} = 1, \quad \frac{\partial M_x}{\partial H_0} = x,$$

and

$$\frac{\partial U}{\partial M_0} = \frac{1}{EI} \int_0^L \left( M_0 - \frac{wl^2}{8} + H_0 x \right) dx = \frac{1}{EI} \left[ M_0 L - \frac{wl^2 L}{8} + \frac{H_0 L^2}{2} \right],$$

while

$$\frac{\partial U}{\partial H_0} = \frac{1}{EI} \int_0^L \left( M_0 - \frac{wl^2}{8} + H_0 x \right) x dx = \frac{1}{EI} \left[ \frac{M_0 L^2}{2} - \frac{wl^2 L^2}{16} + \frac{H_0 L^3}{3} \right].$$

For the half frame therefore,

$$\begin{aligned}M_0 \left( L + \frac{l}{2} \right) - \frac{wl^2}{8} \left( L + \frac{l}{6} \right) + \frac{H_0 L^2}{2} &= 0, \\ \frac{M_0 L}{2} - \frac{wl^2 L}{16} + \frac{H_0 L^2}{3} &= 0,\end{aligned}$$

the solution of which is

$$M_0 = \frac{wl^2}{24} \left\{ \frac{3L+2l}{L+2l} \right\},$$

$$H_0 = \frac{wl^3}{4L(L+2l)},$$

and from symmetry

$$V_0 = \frac{wl}{2}.$$

Part II, Example 2, gives the fully-worked-out case of a polygonal frame with stiff joints under various loadings, and a study of this should enable any problems of this type to be readily treated.

**Degree of Redundancy in an Arch.**—Figure 43 shows an arched rib carried at A and B. To determine the degree of redun-

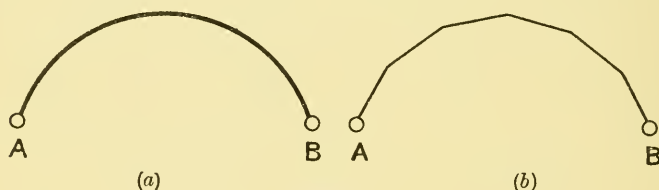


FIG. 43.

dancy imagine the rib to be made up of any number of small straight bars as shown at (b). Suppose that  $N$  points on the rib are connected in this way to the supports A and B. In order that the resulting structure shall be rigidly braced there must be  $2N$  equivalent members. The bars provide  $(N+1)$  of these, and since the joints at each of the  $N$  points are stiff, these furnish  $N$  more. In all, then, there are the equivalent of  $(2N+1)$  members, so that if A and B are pins, there is one redundancy in the structure. This result is independent of  $N$ , so that in the limiting case of the arch, *i.e.* when  $N$  is infinite, there is still one redundancy.

If the arch had been *encastré*, the joints A and B would also provide one equivalent member each, and there would be three redundancies. Fig. 44 will help to make this clear. At (a) is shown a two-pinned arch carrying any load  $W$ . One support must be capable of providing a horizontal reaction to prevent bodily movement of the structure, but if this is provided by fixing A to a pin, say, the support at B need only be capable of furnishing a vertical reaction. It will be seen that if B rests

on a frictionless support, the values of  $V_A$ ,  $V_B$ , and  $H_A$  can be calculated from the conditions of static equilibrium of the arch and the stress distribution is thus statically determinate.

As, however, the point B is also a pinned joint, there is a

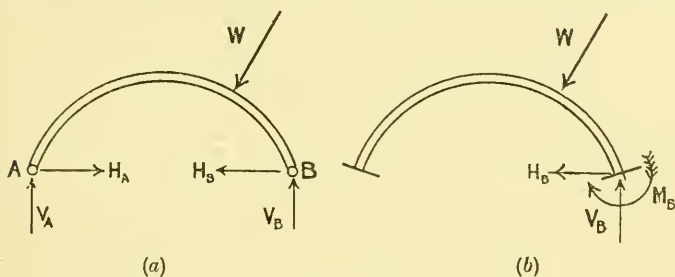


FIG. 44.

horizontal reaction  $H_B$  at B, and the problem can no longer be solved by statics alone.  $H_B$  in this case may conveniently be taken as the redundancy.

If the arch is *encastré* at each end as shown at (b), one support, say at B, can be removed without destroying the structure, which is then a curved cantilever. The support at B, however, provides three reactions—forces  $H_B$  and  $V_B$  and a couple  $M_B$ , and these three reactions may be taken as the redundancies.

Fig. 45 shows a two-pinned arch which will be taken as an

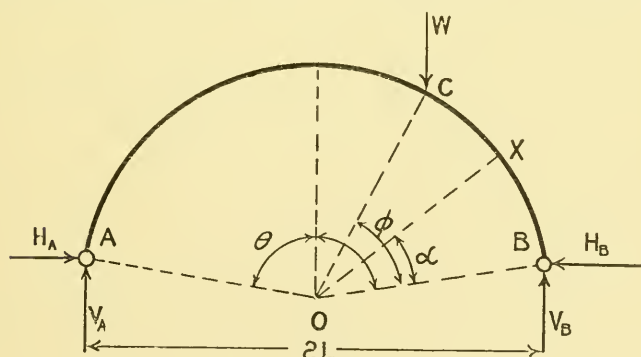


FIG. 45.

example of the procedure in analysing the stresses in a structure of this sort.

The arch is segmental, subtending an angle of  $2\theta$  at the centre of curvature O. It carries a single load  $W$  at a point C, such that  $\angle BOC = \phi$ . The span of the arch is  $2L$  and the radius is  $R$ .

Taking  $H_B$  as the redundancy, we put  $\frac{dU}{dH_B} = 0$ .

From the conditions of static equilibrium of the arch we have

$$\begin{aligned} H_A &= H_B, \\ V_A + V_B &= W, \\ 2LV_B &= W\{L + R \sin(\theta - \phi)\}. \end{aligned}$$

And these give

$$\begin{aligned} V_B &= \frac{\bar{W}}{2L} \{L + R \sin(\theta - \phi)\}, \\ V_A &= \frac{\bar{W}}{2L} \{L - R \sin(\theta - \phi)\}. \end{aligned}$$

At any point X on the rib between B and C such that  $\angle BOX = \alpha$  we have

$$\begin{aligned} M_x &= H_B R \{\cos(\theta - \alpha) - \cos \theta\} - V_B R \{\sin \theta - \sin(\theta - \alpha)\} \\ \frac{dM_x}{dH_B} &= R \{\cos(\theta - \alpha) - \cos \theta\}. \end{aligned}$$

Between B and C, since  $ds = R d\alpha$ ,

$$\frac{dU}{dH_B} = \frac{R^3}{EI} \int_0^\phi [H_B \{\cos(\theta - \alpha) - \cos \theta\}^2 - V_B \{\sin \theta - \sin(\theta - \alpha)\} \{\cos(\theta - \alpha) - \cos \theta\}] d\alpha.$$

And between C and A

$$\begin{aligned} M_x &= H_B R \{\cos(\theta - \alpha) - \cos \theta\} - V_B R \{\sin \theta - \sin(\theta - \alpha)\} \\ &\quad + WR \{\sin(\theta - \phi) - \sin(\theta - \alpha)\}. \end{aligned}$$

$$\text{So} \quad \frac{dM_x}{dH_B} = R \{\cos(\theta - \alpha) - \cos \theta\}$$

and

$$\begin{aligned} \frac{dU}{dH_B} &= \frac{R^3}{EI} \int_\phi^{2\theta} [H_B \{\cos(\theta - \alpha) - \cos \theta\}^2 \\ &\quad - V_B \{\sin \theta - \sin(\theta - \alpha)\} \{\cos(\theta - \alpha) - \cos \theta\} \\ &\quad + \bar{W} \{\sin(\theta - \phi) - \sin(\theta - \alpha)\} \{\cos(\theta - \alpha) - \cos \theta\}] d\alpha. \end{aligned}$$

The equation  $\frac{dU}{dH_B} = 0$  becomes

$$\begin{aligned} \int_0^{2\theta} [H_B \{\cos(\theta - \alpha) - \cos \theta\}^2 \\ - V_B \{\sin \theta - \sin(\theta - \alpha)\} \{\cos(\theta - \alpha) - \cos \theta\}] d\alpha \\ + \int_\phi^{2\theta} [\bar{W} \{\sin(\theta - \phi) - \sin(\theta - \alpha)\} \{\cos(\theta - \alpha) - \cos \theta\}] d\alpha = 0. \end{aligned}$$

Upon integration this gives  $H_B$  in terms of  $V_B$ , which is known from the static equations.

The *encastré* arch is dealt with similarly, except that there are three equations

$$\frac{\partial U}{\partial M_B} = \frac{\partial U}{\partial H_B} = \frac{\partial U}{\partial V_B} = 0.$$

The general algebraic solution of this problem is somewhat complicated, but in a numerical example the work is straightforward.

## CHAPTER IV

### THE DIRECT DESIGN OF REDUNDANT FRAMEWORKS

IN the design of any frame it is first of all necessary to determine the external load system to which the frame is to be subjected. This being done, the internal forces induced by the external loading must be calculated, and the sizes of the members so proportioned that they will take these forces without being unduly stressed.

In the case of a just-stiff frame this is a simple matter since the forces in the bars depend only on the external loads and the geometry of the frame, but if redundant members are present, the internal forces depend upon the elastic properties of the members as well as upon the other conditions.

A difficulty is at once apparent—the stress distribution depends upon the size of the bars, which cannot be known until the stresses they have to carry are found. This can be overcome by a series of approximations, and the general procedure would be to guess approximate sizes for all members of the truss and then analyse the stresses by the methods described in the previous chapters. If the loads are found to be just such as the assumed members can carry, the design is satisfactory, but in general some of the bars will be found to be too small, in which case the frame is not safe; or too large, in which case the structure is uneconomical. In either event a second attempt is necessary and the whole laborious process must be repeated, using the first approximation as a basis for a closer guess.

In this chapter, a method is described<sup>1</sup> which eliminates the trial and error of successive approximations, and enables a frame to be designed directly, the designer controlling his stresses as the work proceeds. It is necessary to emphasize that the design produced in this way will be in the nature of an “ideal” design which may, and probably will, require some modification due to

<sup>1</sup> “On a Method for the Direct Design of Framed Structures having Redundant Bracing.” A. J. S. Pippard. Reports and Memoranda No. 793. Aeronautical Research Committee, H.M. Stationery Office.

practical considerations, but it will afford a sound basis from which a practical design can be obtained with the minimum of trouble.

Suppose, then, that the frame to be designed carries a number of external loads denoted by  $P_1, P_2, P_3, \dots P_N$ , and that it contains a number of redundant members, the forces in which we will call  $R_1, R_2, \dots R_M$ . In the usual way, by drawing a number of separate stress diagrams, the loads in all members of the truss can be found in terms of the external loads and the redundant forces, in the form

$$P_0 = \{aP_1 + bP_2 + \dots + nP_N + \alpha R_1 + \beta R_2 + \dots + \mu R_M\}$$

where  $a, b, \dots n$  and  $\alpha, \beta, \dots \mu$  are simply numerical coefficients depending on the geometry of the frame.

The strain energy of the whole structure is

$$U = \sum \frac{1}{2} \cdot \frac{P_0^2 L}{AE}$$

and by the principle of least work we can write

$$\frac{\partial U}{\partial R_1} = \frac{\partial U}{\partial R_2} = \dots = \frac{\partial U}{\partial R_M} = 0.$$

Now 
$$\frac{\partial U}{\partial R_1} = \sum \frac{P_0 L}{AE} \cdot \alpha,$$

where  $\alpha$  is the coefficient of  $R_1$  in the load in any member, and  $\frac{P_0}{A} = f$  is the stress in the material of the member,

hence 
$$\left. \begin{aligned} \frac{\partial U}{\partial R_1} &= \sum \frac{fL\alpha}{E} = 0, \\ \frac{\partial U}{\partial R_2} &= \sum \frac{fL\beta}{E} = 0, \\ &\dots \dots \dots \\ \frac{\partial U}{\partial R_M} &= \sum \frac{fL\mu}{E} = 0. \end{aligned} \right\} \dots \dots \dots (23)$$

If, as is usual, the material of the frame is the same throughout,  $E$  is constant and the equations become

$$\sum fL\alpha = \sum fL\beta = \dots = \sum fL\mu = 0 \quad \dots \dots (24)$$

This result may be used for design purposes as follows:—

Replace all redundant members in the structure by unknown forces  $R_1, R_2$ , etc., acting along the axes of these members.

Since each member is connected to two joints of the framework, the force replacing it must be applied at both joints.

The structure is now reduced to a just-stiff framework acted upon by the external load system  $P_1, P_2$ , etc., and the forces  $R_1, R_2$ , etc. A stress diagram is drawn for the external load system, and the forces in all members due to this system are tabulated. This gives the terms  $aP_1 + bP_2$ , etc., in the expression for  $P_0$ . Next draw a stress diagram for the two forces of magnitude  $R_1$ —one at each joint connected by the member concerned—and tabulate the internal forces in all members due to these. This diagram gives the terms  $\alpha R_1$  for each member of the truss. Similar diagrams are drawn for  $R_2, R_3 \dots R_M$  acting separately, and these give the coefficients  $\beta, \gamma \dots \mu$ . Hence the coefficients  $\alpha, \beta \dots \mu$  are determined for all members of the framework, and also the loads in them in terms of  $P_1, P_2 \dots R_1, R_2 \dots$  etc. The length of each member in the structure is given by the frame diagram and the material to be used in construction is known. Hence the values of  $L\alpha, L\beta$ , etc., if the material is the same throughout the structure, or  $\frac{L\alpha}{E}, \frac{L\beta}{E}$ , etc., if the material varies, can be calculated for each member.

The series of equations corresponding to equation (23) or equation (24) can now be written down: there are the same number as there are redundant members. Each of these equations contains terms involving the stress in one of the redundant members and the stresses in those members of the structure which are affected by that particular redundancy. No other terms occur. Thus the equations connect the stresses in the various members of the structure, and the next step is to select such stresses as will satisfy the equations.

It will be found convenient to start with the equation which contains the smallest number of terms. The maximum permissible value of the stress for the material can be substituted for the majority of terms occurring, the remainder—which need not be more than one—being adjusted to satisfy the equation. This will be found a very easy matter since there are any number of possible variations, all correct, and it is not a question of determining a unique solution.

Certain stresses which have been fixed occur in other equations of the series, and these should be substituted in their proper places. The remaining equations can then be dealt with in the same way and the stresses in all members of the structure will then be fixed. It should be pointed out that these equations have not to be solved simultaneously, but can be taken almost

independently, and the work involved in obtaining reasonable solutions is very light.

The stresses fixed for each member of the structure should be tabulated, and among these, of course, occur the stresses in each of the redundant members. The next step is to fix the magnitude of the load in each redundancy by giving a suitable area to these members. A study of the table of internal loads enables a decision to be made as to what is a suitable magnitude for the forces in the redundant members by showing the effect these loads have upon other members of the structure. The determined values divided by the stresses already fixed give the cross-sectional areas. Once the sizes of the redundant members have been fixed, the absolute values of the loads in all the other members of the structure can be written down,

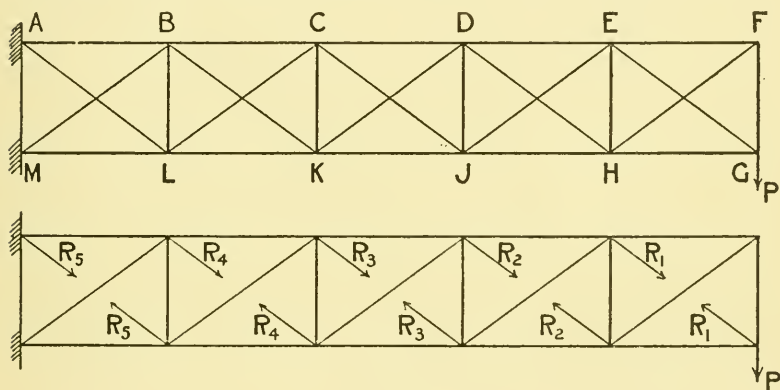


FIG. 46.

and as the stress to be developed in each of these members is already known, the determination of cross-sectional areas becomes simply a matter of division. Thus by a direct process the cross-sectional areas of all members of the structure are designed.

As an example of the foregoing method the double-braced cantilever shown in Fig. 46 will be considered. This structure is taken purely as an example illustrative of the method, and has been selected since it has a comparatively large number of redundancies. It is supported at the points A and M, and a load  $P$  is carried at the point G. The magnitude of this load will be known, and for the present calculation has been taken to be 3,000 lbs. The structure is everywhere pin-jointed, and the modulus of elasticity is the same throughout.

One of each pair of diagonal bracing members is redundant, and EG, DH, CJ, BK and AL will be taken as the redundant members. The unknown forces in these members will be denoted by  $R_1, R_2, R_3, R_4$  and  $R_5$ . Eliminating these members, the forces in the remaining members of the truss under the action of  $P$  can be determined by the resolution of forces at each joint. Inserting  $R_1, R_2$ , etc., in turn, the forces in all members due to these unknown loads can be determined, and these are given in Table I, col. 2. This table is so arranged that the values of  $\alpha, \beta$ , etc., are conveniently presented, and since the length of each member is known, we can write down equations of the form

$$fLa=0, \text{ etc.}$$

Denoting the stress in any member AB by  $\widehat{AB}$  the equations become

$$-1.8\widehat{FG}-3.2\widehat{EF}+3.2\widehat{HG}+5\widehat{EG}-5\widehat{FH}-1.8\widehat{EH}=0 \quad . \quad . \quad (25)$$

$$-1.8\widehat{EH}-3.2\widehat{ED}+3.2\widehat{JH}+5\widehat{DH}-5\widehat{EJ}-1.8\widehat{JD}=0 \quad . \quad . \quad (26)$$

$$-1.8\widehat{JD}-3.2\widehat{CD}+3.2\widehat{KJ}+5\widehat{CJ}-5\widehat{KD}-1.8\widehat{CK}=0 \quad . \quad . \quad (27)$$

$$-1.8\widehat{CK}-3.2\widehat{BC}+3.2\widehat{LK}+5\widehat{BK}-5\widehat{CL}-1.8\widehat{BL}=0 \quad . \quad . \quad (28)$$

$$-1.8\widehat{BL}-3.2\widehat{AB}+3.2\widehat{ML}+5\widehat{AL}-5\widehat{MB}=0 \quad . \quad . \quad (29)$$

It will be assumed that the maximum stress in a tension member must not exceed 10,000 lbs. per square inch, and in a compression member 6,000 lbs. per square inch.

Starting with equation (29), put

$$\widehat{AB}=10,000 \text{ lbs. per square inch}$$

$$\widehat{ML}= 8,000 \quad , \quad ,$$

$$\widehat{MB}= 8,000 \quad , \quad ,$$

Substitute these values and we get

$$-1.8\widehat{BL}+5\widehat{AL}=46,400.$$

$$\text{Make} \quad \widehat{AL}=10,000 \text{ lbs. per square inch}$$

$$\text{and} \quad \widehat{BL}= 2,000 \quad , \quad ,$$

and equation (29) is satisfied.

Now consider equation (28).  $\widehat{BL}$  has been fixed at 2,000 lbs. per square inch.

Make	$\widehat{BC}=10,000$ lbs. per square inch
	$\widehat{LK}= 8,000$ „     „
	$\widehat{BK}=10,000$ „     „
	$\widehat{CK}= 2,000$ „     „
This gives	$\widehat{CL}= 7,280$ „     „

The remainder of the equations are of the same form as (28), and the stresses in corresponding members can therefore be made the same as those just determined. The stresses are given in Table I, col. 3.

The next step is to fix the absolute values of the loads in all the redundant members.

The loads in the vertical members have been assumed to be tensile, and the values of  $R_1$ ,  $R_2$ , etc., should be so selected that this is ensured, as otherwise certain signs in the stress equations will require modification.

Make  $R_1=R_2=R_3=R_4=R_5=2,000$  lbs.

Since  $P$  is 3,000 lbs. the loads in all members of the truss can now be obtained by substitution in Table I, col. 1, and the absolute values of the loads are given in col. 4.

These loads divided by the stresses already determined fix the necessary main sizes of the design.

To show the relative amounts of work involved in the present method and the usual application of the principle of least work the example just dealt with is also treated by the latter method. In order that the two results may agree, the areas already fixed in Table I will be used in the strain energy calculations, although it must be remembered that in an actual design started *ab initio* these would have to be guessed.

In the first place, the loads in all members must be found in terms of the known external loading and the unknown loads in redundant members. This work is common to both methods, and the results are given in Table I, col. 1. The remainder of the work is most satisfactorily done in a tabular manner, and is shown in Table II. The first four columns are self-explanatory; the fifth gives the load in each member taken from Table I, coefficients only being tabulated for ease of manipulation.

Col. 6 gives the value of  $\frac{P_0 L}{A} \frac{\partial P_0}{\partial R_1}$  for each member, coefficients again being used.

Since  $\frac{P_0 L}{AE} \frac{\partial P_0}{\partial R_1}$  is the value of  $\frac{\partial U}{\partial R_1}$ , the expressions in col. 6

are proportional to the differential coefficient of the strain energy in each member with respect to the redundant force  $R_1$ . The value of  $E$ , being constant throughout the structure, has been omitted.

The sum of these expressions for all individual members is proportional to  $\frac{\partial U}{\partial R_1}$ , and by the principle of least work this is equal to zero.

Hence by adding the various terms of col. 6 and equating to zero, the first equation is obtained. Cols. 7-10 give the same results for  $\frac{\partial U}{\partial R_2}$ , etc., and provide the remaining equations. These five equations must be solved simultaneously to determine the values of the forces in the redundant members.

The equations thus obtained for this particular example are

$$\left. \begin{array}{l} 65.40R_1 + 3.60R_2 \qquad \qquad \qquad -46.01P = 0 \\ 3.60R_1 + 51.99R_2 + 3.60R_3 \qquad \qquad -39.53P = 0 \\ \qquad 3.60R_2 + 48.93R_3 + 3.60R_4 \qquad -37.47P = 0 \\ \qquad \qquad 3.60R_3 + 47.62R_4 + 3.60R_5 - 36.62P = 0 \\ \qquad \qquad \qquad 3.60R_4 + 44.49R_5 - 32.03P = 0 \end{array} \right\}$$

and the solution of these simultaneous equations is

$$R_1 = R_2 = R_3 = R_4 = R_5 = \frac{2}{3}P.$$

Since  $P$  is 3,000 lbs., the force in each redundant bar is 2,000 lbs. as fixed when designing the structure by the method of this chapter.

It will be realized from a comparison of the two examples given that a considerable saving of time and work can be gained by the use of the direct method, and this saving will be increased if, as usually happens, the work in the second instance has to be done more than once on account of injudicious selection of areas in the beginning.

TABLE I.

1	2						3	4	5
Member.	Load Coefficients.						Stress.	Load.	Area.
	P	R <sub>1</sub>	R <sub>2</sub>	R <sub>3</sub>	R <sub>4</sub>	R <sub>5</sub>	lbs. per sq. in.	lbs.	sq. in.
AB	6.67	—	—	—	—	—0.8	10,000	18,400	1.84
ML	5.33	—	—	—	—	0.8	8,000	17,600	2.2
BM	1.67	—	—	—	—	—1.0	8,000	3,000	0.375
AL	—	—	—	—	—	1.0	10,000	2,000	0.200
BL	1.0	—	—	—	—0.6	—0.6	2,000	600	0.300
CK	1.0	—	—	—0.6	—0.6	—	2,000	600	0.300
CL	1.67	—	—	—	—1.0	—	7,280	3,000	0.412
BC	5.33	—	—	—	—0.8	—	10,000	14,400	1.44
LK	4.0	—	—	—	0.8	—	8,000	13,600	1.70
BK	—	—	—	—	1.0	—	10,000	2,000	0.200
JD	1.0	—	—0.6	—0.6	—	—	2,000	600	0.300
CD	4.0	—	—	—0.8	—	—	10,000	10,400	1.04
KJ	2.67	—	—	0.8	—	—	8,000	9,600	1.20
CJ	—	—	—	1.0	—	—	10,000	2,000	0.20
KD	1.67	—	—	—1.0	—	—	7,280	3,000	0.412
EH	1.0	—0.6	—0.6	—	—	—	2,000	600	0.300
ED	2.67	—	—0.8	—	—	—	10,000	6,400	0.64
JH	1.33	—	0.8	—	—	—	8,000	5,600	0.70
DH	—	—	1.0	—	—	—	10,000	2,000	0.20
EJ	1.67	—	—1.0	—	—	—	7,280	3,000	0.412
FG	1.0	—0.6	—	—	—	—	2,000	1,800	0.90
EF	1.33	—0.8	—	—	—	—	10,000	2,400	0.24
HG	—	0.8	—	—	—	—	8,000	1,600	0.20
FH	1.67	—1.0	—	—	—	—	7,280	3,000	0.412
EG	—	1.0	—	—	—	—	10,000	2,000	0.200



TABLE II (continued).

STRAIN ENERGY CALCULATIONS.

1	2	3	4	5						9						10					
				P <sub>0</sub>						$\frac{P_0 L}{A} \frac{\partial_0 P}{\partial R_4}$						$\frac{P_0 L}{A} \frac{\partial_0 P}{\partial R_5}$					
				R <sub>1</sub>	R <sub>2</sub>	R <sub>3</sub>	R <sub>4</sub>	R <sub>5</sub>	P	R <sub>1</sub>	R <sub>2</sub>	R <sub>3</sub>	R <sub>4</sub>	R <sub>5</sub>	P	R <sub>1</sub>	R <sub>2</sub>	R <sub>3</sub>	R <sub>4</sub>	R <sub>5</sub>	P
AB	4	1.84	30	—	—	—	—	-0.8	6.67	—	—	—	—	—	—	—	—	—	—	—	-11.58
BC	4	1.44	"	—	—	—	-0.8	—	5.33	—	—	—	1.87	—	-11.78	—	—	—	—	1.39	—
CD	4	1.04	"	—	—	-0.8	—	—	4.0	—	—	—	—	—	—	—	—	—	—	—	—
DE	4	0.64	"	—	-0.8	—	—	—	2.67	—	—	—	—	—	—	—	—	—	—	—	—
EF	4	0.24	"	-0.8	—	—	—	—	1.33	—	—	—	—	—	—	—	—	—	—	—	—
ML	4	2.20	"	—	—	—	—	—	5.33	—	—	—	—	—	—	—	—	—	—	1.17	7.75
LK	4	1.70	"	—	—	—	—	—	4.0	—	—	—	1.51	—	7.55	—	—	—	—	—	—
KJ	4	1.20	"	—	—	0.8	—	—	2.67	—	—	—	—	—	—	—	—	—	—	—	—
JH	4	0.70	"	—	0.8	—	—	—	1.33	—	—	—	—	—	—	—	—	—	—	—	—
HG	4	0.20	"	0.8	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
FG	3	0.90	"	-0.6	—	—	—	—	1.0	—	—	—	—	—	—	—	—	—	—	—	—
EH	3	0.30	"	-0.6	-0.6	—	—	—	1.0	—	—	—	—	—	—	—	—	—	—	—	—
DJ	3	0.30	"	—	-0.6	-0.6	—	—	1.0	—	—	3.60	3.60	—	-6.00	—	—	—	—	—	—
CK	3	0.30	"	—	—	—	-0.6	-0.6	1.0	—	—	—	3.60	3.60	-6.00	—	—	—	3.60	3.60	-6.00
BL	3	0.30	"	—	—	—	—	—	1.0	—	—	—	—	—	—	—	—	—	—	—	—
EG	5	0.20	"	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
PH	5	0.412	"	1.0	—	—	—	—	1.67	—	—	—	—	—	—	—	—	—	—	—	—
DH	5	0.20	"	-1.0	1.0	—	—	—	1.67	—	—	—	—	—	—	—	—	—	—	—	—
EJ	5	0.412	"	—	-1.0	—	—	—	1.67	—	—	—	—	—	—	—	—	—	—	—	—
CJ	5	0.20	"	—	—	1.0	—	—	1.67	—	—	—	—	—	—	—	—	—	—	—	—
DK	5	0.412	"	—	—	-1.0	—	—	1.67	—	—	—	—	—	—	—	—	—	—	—	—
BK	5	0.20	"	—	—	—	—	—	1.67	—	—	—	—	—	—	—	—	—	—	—	—
CL	5	0.412	"	—	—	—	—	—	1.67	—	—	—	—	—	-20.3	—	—	—	—	—	—
AL	5	0.20	"	—	—	—	—	—	1.67	—	—	—	—	—	—	—	—	—	—	—	—
EM	5	0.375	"	—	—	—	—	-1.0	1.67	—	—	—	—	—	—	25.0	—	—	—	25.0	-22.2
				—	—	—	—	-1.0	1.67	—	—	—	—	—	—	13.33	—	—	—	13.33	—

## PART II

### EXAMPLE 1

#### THE STRESSES IN AN AEROPLANE FUSELAGE UNDER TORSIONAL LOADS

THIS example gives the treatment of a comparatively simple framework, and shows in detail the procedure of analysis and the method of tabulation. The fuselage of an aeroplane may be considered to be held firmly at the wings by the centre-section structure. Aerodynamic loads from the tail-plane, fin and rudder are applied to the end of the fuselage, and since these loads are often offset from the axis of the aeroplane, the fuselage is subjected to a torque.

The structure of a fuselage of the type considered in this example consists of four longitudinal members or longerons divided into panels by vertical and horizontal struts. Each of these panels is braced and counterbraced by swaged rods or wires. The "bulkheads" formed by the four struts at each cross-section are also braced and counterbraced in the same way. It is generally assumed that the structure is pin-jointed throughout.

If the bracing wires are initially tensioned, they can, if required, take a certain amount of compressive load, but if put in without such initial tension, the wires called upon to carry compression slacken and the frame is in fact pseudo-redundant. In this example the latter case will be assumed and one wire only of each pair will be considered to be operative.

This treatment was first published by the Aeronautical Research Committee.<sup>1</sup>

Fig. 47 shows a fuselage of the type considered. It is supposed to be held rigidly by the centre section of the aeroplane at the face ABCD, and the torque of magnitude  $\mu$  is applied to the face EFHG, which may also be assumed to be rigid in its own plane on account of the bracing of the tail structure (not shown

<sup>1</sup> Reports and Memoranda No. 736. "Torsional Stresses in the Fuselage of an Aeroplane." A. J. S. Pippard and W. D. Douglas.

in the diagram). All four faces BEGC, ABEF, ADHF, and DCGH are plane, and each section of the fuselage is rectangular.

The structure shown has sixteen nodes to be braced to the four fixed points A, B, C and D, and in order to make a just-stiff frame forty-eight members are needed. Hence if the fuselage were erected without any bulkhead bracing wires, it would be just-stiff. The end face, however, is supposed to be rigid in its own plane, so that the four points composing it may be assumed to be connected together by five inextensible members. Since this is one more than the stiffness of the fuselage needs, the structure without bulkhead wires, but with a rigid end face, is redundant to the extent of one bar. When the face EFHG is loaded by the torque, shear forces are applied to the four faces

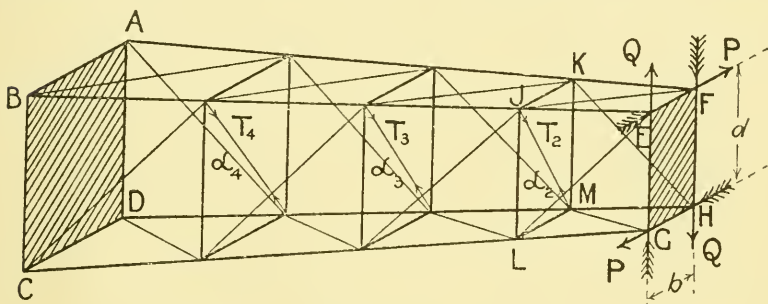


FIG. 47.

of the fuselage as shown. These form two couples,  $Pd$  and  $Qb$ , which together are equivalent to the torque, so that

$$Pd + Qb = \mu.$$

One of these unknown forces can be taken as the redundancy, and the strain energy of the frame can be obtained in terms of it. Then, taking  $P$  as the redundant force, we can write

$$\frac{dU}{dP} = 0.$$

If now the frame is provided with counterbraced bulkhead wires, with no initial tensions, we introduce one redundancy at each bulkhead. In the example shown in Fig. 47, there are three from this cause shown as  $T_2$ ,  $T_3$ , and  $T_4$ , so that this structure has four redundancies in all. It is not easy to judge which one of a pair of bulkhead wires will be the operative or tension member, and if a wrong assumption is made it will be shown in the final result by a negative sign attached to the load

in that member. If this load were large, the calculations would have to be repeated with the correct wire taken.

In the particular problem now being investigated however, the loads in these bulkhead wires will be shown to be so small that their effects are negligible, and so it is immaterial which one is assumed to be operative.

In any case the method is unaffected by this detail. It will be assumed that the diagonal wires shown in each bulkhead are in tension, and are replaced by external loads  $T_2$ ,  $T_3$ , etc., acting at the appropriate nodes.

Let the angles between the bulkhead wires and the vertical be denoted by  $\alpha_2$ ,  $\alpha_3$ , and  $\alpha_4$ . The force  $T_2$  acting at J can be resolved into two components,  $T_2 \sin \alpha_2$  acting along JK, and

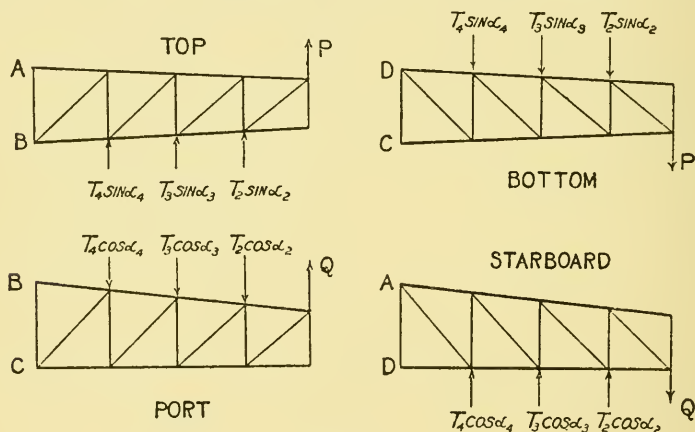


FIG. 48.

$T_2 \cos \alpha_2$  acting along JL. Similarly, the force  $T_2$  acting at M can be resolved into  $T_2 \sin \alpha_2$  acting along ML, and  $T_2 \cos \alpha_2$  acting along MK. The other bulkheads may be treated in the same way, and the four faces of the fuselage are then loaded as shown in Fig. 48, the unknown forces which have to be determined being P, Q,  $T_2$ , etc. By drawing stress diagrams or by the use of the method of sections, the load in each member of the structure can be found in terms of these unknown forces. The stresses in the struts and wires involve only the loads acting in their own plane, but since each longeron forms the boom for two faces of the fuselage, the stress in each section of the longeron involves loads acting on two faces of the structure. The total load in each bay of the longeron is the algebraic sum of the stresses arising from the two appropriate load systems.

The total energy stored in the fuselage is the sum of the strain energy of the various members

$$U = \Sigma u = \Sigma \frac{P_0^2 L}{2AE}.$$

Now the load  $P_0$  in any member of the fuselage contains some or all of the terms  $P, Q, T_2 \dots T_n$ , of which  $P$  and  $Q$  are related variables, so that  $Q$  can be expressed in terms of  $P$  and the known values of  $\mu, d$ , and  $b$ . Hence the most general form of the expression for the load occurring in a member is

$$P_0 = (AP + B\mu + DT_2 \dots KT_n),$$

where  $A, B \dots K$  are numerical coefficients.

The values of  $P, T_2$ , etc., are such as cause the structure to store the minimum strain energy, so that the following conditions must be satisfied:—

$$\begin{aligned} \frac{\partial U}{\partial P} &= 0 \\ \frac{\partial U}{\partial T_2} &= 0 \\ \frac{\partial U}{\partial T_3} &= 0, \text{ etc.} \end{aligned}$$

There is one equation of this series for each unknown, thus giving sufficient for a complete solution of the problem.

Having determined the load occurring in each member of the truss in terms of the unknown forces, the expression for the total strain energy can be obtained provided that the cross-sectional areas of the members are known.

The actual example to be worked out is shown in Fig. 49.

Making use of the fact that

$$\frac{\partial U}{\partial T_2} = \Sigma \frac{P_0 L}{AE} \frac{\partial P_0}{\partial T_2}, \text{ etc.,}$$

the work can be carried out in tabular form and an example is given on p. 91 for the panel wires. Col. 1 indicates the member, cols. 2 and 3 its length and cross-sectional area respectively.

In col. 4 the coefficients of  $Q, P$ , and  $T_2 \dots T_4$  in the load  $P_0$  are given, and col. 5 is similar, with  $Q$  eliminated by writing  $Q = \frac{\mu - Pd}{b}$ .

Columns now follow for  $\frac{P_0 L}{AE} \frac{\partial P_0}{\partial T_2}, \frac{P_0 L}{AE} \frac{\partial P_0}{\partial T_3}$  and  $\frac{P_0 L}{AE} \frac{\partial P_0}{\partial T_4}$ , of

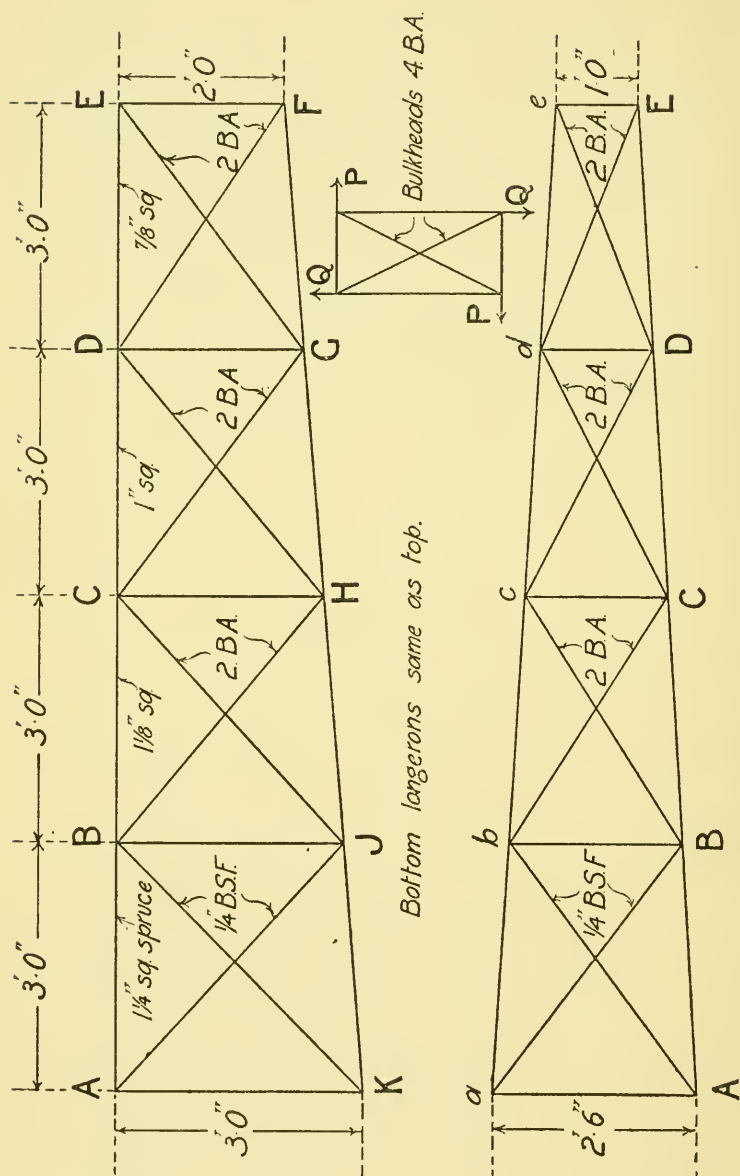


FIG. 49.

1	2	3	4		5				6		7						
Member.	Length L.	Area A.	Load P <sub>0</sub> .				Load (eliminating Q).				$\frac{P_0 L}{AE} \frac{\partial P_0}{\partial T_2}$ .		$\frac{P_0 L}{AE} \frac{\partial P_0}{\partial T_3}$ .				
			Coefficients.				Coefficients.				Coefficients × 10 <sup>6</sup> .		Coefficients × 10 <sup>6</sup> .				
	Ins.	Sq. in.	Q	P	T <sub>2</sub>	T <sub>3</sub>	T <sub>4</sub>	μ	P	T <sub>2</sub>	T <sub>3</sub>	T <sub>4</sub>	μ	P	T <sub>2</sub>	T <sub>3</sub>	T <sub>4</sub>
Port panel wires.	49.4	0.023	-1.0	—	+0.955	+1.02	+1.095	-1.0	+2.0	+0.955	+1.02	+1.095	-1.0	73	146	70	74
	48.6	0.0129	-1.18	—	+1.125	+1.21	—	-1.18	+2.36	+1.125	+1.21	—	-167	334	159	171	184
	46.8	0.0129	-1.38	—	+1.32	—	—	-1.38	+2.76	+1.32	—	—	-220	440	211	—	—
	44.4	0.0129	-1.65	—	—	—	—	-1.65	+3.30	—	—	—	—	—	—	—	—
Starboard panel wires.	48.5	0.023	-1.0	—	+0.94	+1.0	+1.07	-1.0	+2.0	+0.94	+1.0	+1.07	-1.0	66	132	62	66
	46.6	0.0129	-1.15	—	+1.08	+1.16	—	-1.15	+2.3	+1.08	+1.16	—	-149	299	140	151	162
	45.6	0.0129	-1.35	—	+1.28	—	—	-1.35	+2.7	+1.28	—	—	-204	408	193	—	—
	43.2	0.0129	-1.60	—	—	—	—	-1.6	+3.2	—	—	—	—	—	—	—	—
Top panel wires.	45.4	0.023	—	-0.74	-0.51	-0.713	-0.93	—	-0.74	-0.51	-0.713	-0.93	—	26	17	24	31
	43.0	0.0129	—	-0.96	-0.69	-0.965	—	—	-0.96	-0.69	-0.965	—	—	72	52	73	—
	40.8	0.0129	—	-1.41	-1.02	—	—	—	-1.41	-1.02	—	—	—	152	109	—	—
	38.9	0.0129	—	-2.35	—	—	—	—	-2.35	—	—	—	—	—	—	—	—
Bottom panel wires.	45.4	0.023	—	-0.74	-0.51	-0.713	-0.93	—	-0.74	-0.51	-0.713	-0.93	—	26	17	24	31
	43.0	0.0129	—	-0.96	-0.69	-0.965	—	—	-0.96	-0.69	-0.965	—	—	72	52	73	—
	40.8	0.0129	—	-1.41	-1.02	—	—	—	-1.41	-1.02	—	—	—	152	109	—	—
	38.9	0.0129	—	-2.35	—	—	—	—	-2.35	—	—	—	—	—	—	—	—
Bulkhead wires.	41.7	0.0085	—	—	—	—	-1.0	—	—	—	—	-1.0	—	—	—	—	—
	36.6	0.0085	—	—	—	-1.0	—	—	—	—	-1.0	—	—	—	—	—	143
	31.65	0.0085	—	—	-1.0	—	—	—	—	-1.0	—	—	—	—	—	—	—
	26.8	0.0085	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
								-874	2249	1310	652	207	-483	1238	652	905	241

which two only are shown. Since  $\frac{\partial P_0}{\partial T_3}$ , etc., are the coefficients of  $T_3$ , etc., as given in col. (5), it is a simple matter to write down the values in col. 6 *et seq.*

Finally, the values of  $\sum_{AE} \frac{P_0 L}{\partial T_2}$ , etc., are found by summing cols. 6, 7, etc.

The summation of these columns provides the necessary equations for the complete solution of the problem; in the present case these are

$$\begin{aligned} 27270P + 7365T_2 + 3496T_3 + 927T_4 &= 8924\mu \\ 12166P + 3291T_2 + 1564T_3 + 414T_4 &= 3980\mu \\ 7363P + 2822T_2 + 1380T_3 + 385T_4 &= 2436\mu \\ 3494P + 1382T_2 + 1319T_3 + 355T_4 &= 1177\mu \\ 928P + 388T_2 + 357T_3 + 482T_4 &= 315\mu. \end{aligned}$$

The solution of these is

$$\begin{aligned} P &= 0.321\mu \\ Q &= 0.358\mu \\ T_2 &= 0.009\mu \\ T_3 &= 0.028\mu \\ T_4 &= 0.010\mu. \end{aligned}$$

Substituting these in the values for  $P_0$ , the loads in all members of the frame are found.

Giving  $\mu$  the value 1,000 ft.-lbs. these loads are as set out in the table following :

Member.	Load. Lbs.	Member.	Load. Lbs.
KB	-310	AB ; kj	- 46
JC	-400	BC ; jh	15
HD	-482	CD ; hg	151
GE	-590	DE ; gf	476
aj	-311	KJ ; ab	485
bh	-370	JH ; bc	551
cg	-473	HG ; cd	632
df	-572	GF ; de	700
Ab ; kJ	-271	BJ ; bj	231
Bc ; jH	-341	CH ; ch	279
Cd ; hG	-460	DG ; dg	318
De ; gF	-754	Bb ; Jj	176
jB	- 10	Cc ; Hh	206
hC	- 28	Dd ; Gg	240
gD	- 9		

## EXAMPLE 2

### THE STRESSES IN A STIFF-JOINTED POLYGONAL FRAME

IN connection with certain investigations concerned with the stresses in the hulls of rigid airships it became necessary to determine the stresses in a polygonal frame which relied for its bracing merely on the stiffness of the joints. This example shows the treatment adopted.

The original work was published by the Aeronautical Research Committee in a series of three papers.<sup>1</sup>

#### SECTION I.—LOADS APPLIED IN PLANE OF FRAME PARALLEL TO ITS AXIS OF SYMMETRY.

If a plane polygonal framework with stiff joints be acted upon by an external load system in its own plane, the resultant actions at any section of the frame consist of a bending moment, a force normal to the section and a shear parallel to the section. In order to obtain the stresses in such a framework it is necessary to determine these resultant actions for all sections. The framework considered is dependent for its resistance to collapse only upon the stiffness of the joints. The flexural rigidity  $EI$  is taken as constant throughout the frame, which is in equilibrium under the action of external loads arranged symmetrically about an axis through the centre of the circle circumscribing the polygon. The loads are applied at the joints and their lines of action are parallel to the axis of symmetry (Fig. 50).

The two cases of a regular polygon having an even and an odd number of sides are treated separately.

For the purpose of analysis the external load system may be divided into a number of pairs of loads, each pair being symmetrical with respect to the central axis. The effect upon the structure

<sup>1</sup> Reports and Memoranda of the Aeronautical Research Committee, Nos. 820 and 912, "Stresses in a Stiff-jointed Polygonal Frame under Loads in its own Plane," A. J. S. Pippard; and No. 1039, "Stresses in a Stiff-jointed Polygonal Frame under a System of Loads Perpendicular to the Plane of the Frame," J. F. Baker.

of each separate pair having been found, the principle of superposition can be used to obtain the net effect of all the loads acting

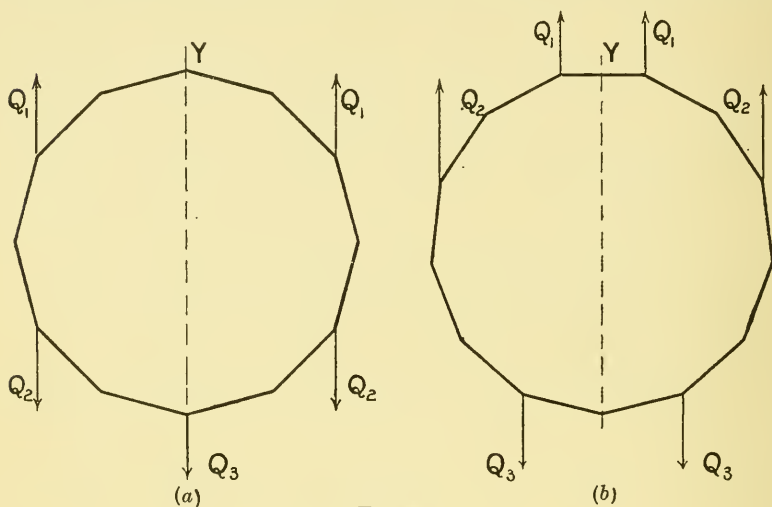


FIG. 50.

simultaneously. Imagine the framework to be cut through at the point Y on the axis of symmetry (Fig. 51). In order to

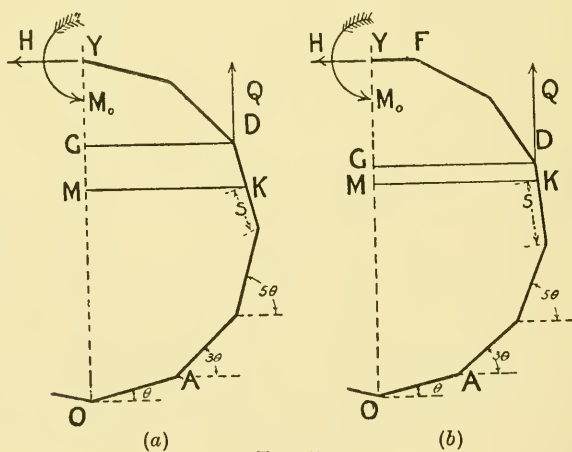


FIG. 51.

maintain the conditions which exist in the absence of such a cut, it is necessary to apply a couple and a horizontal force to each of the cut sections. Considering the half structure shown in Fig. 51,

let the couple applied to the section at Y be  $M_0$  and let the horizontal force be  $H$ . From considerations of the symmetry of the complete loaded structure it is evident that there is no horizontal movement and no angular rotation of the section Y. The values of  $M_0$  and  $H$  must, therefore, be such as to satisfy these conditions.

If  $U$  be the total strain energy of the structure, we may write by the first theorem of Castigliano

$$\text{angular rotation of section} = \frac{\partial U}{\partial M_0} = 0,$$

$$\text{horizontal movement of section} = \frac{\partial U}{\partial H} = 0.$$

It has been demonstrated by various writers <sup>1</sup> that in problems analogous to the one under consideration, *e.g.* in the case of an arched rib or a circular ring, the effect of including the strain energy due to thrust and shear is negligible.<sup>2</sup> In what follows, therefore, the strain energy due to bending is alone considered.

If  $M$  is the bending moment at any point, the strain energy due to bending is given by

$$U = \frac{1}{2} \int \frac{M^2}{EI} ds.$$

$$\text{Hence} \quad \frac{\partial U}{\partial M_0} = \frac{1}{EI} \int M \frac{\partial M}{\partial M_0} ds,$$

$$\text{and} \quad \frac{\partial U}{\partial H} = \frac{1}{EI} \int M \frac{\partial M}{\partial H} ds.$$

In Fig. 51 take the point O at the bottom of the framework as the origin and let a load  $Q$  be applied at the end of the  $k$ th side from the origin.

The moment of any section will be considered positive when it tends to rotate that part of the structure of Fig. 51 which lies between the origin and the section in question in a counter-clockwise direction about the origin, and forces will be considered positive when they tend to produce this counter-clockwise movement.

Then the bending moment at any point K on the  $r$ th side from the origin is given by

$$M_K = M_0 + MY \cdot H - (MK - GD)Q.$$

<sup>1</sup> Cp. Church, *Mechanics of Internal Work*, pp. 101 *et seq.*

<sup>2</sup> See also Part I, p. 39.

**Case I.—Polygon with an even number of sides.**—This case is shown at (a) in Figs. 50 and 51.

Let the polygon have  $2n$  sides and let the angle subtended at the centre by a side be  $2\theta$ , so that  $\theta = \frac{\pi}{2n}$ .

Let the distance of the point K, measured from the beginning of the  $r$ th side, be  $s$ . Then

$$OM = OA\{\sin \theta + \sin 3\theta + \sin 5\theta \dots \sin (2r-3)\theta\} + s \sin (2r-1)\theta.$$

If  $R$  be the radius of the circumscribing circle to the polygon,

$$OA = 2R \sin \theta$$

$$\text{and} \quad 2R - OM = R\{1 + \cos 2(r-1)\theta\} - s \sin (2r-1)\theta.$$

Also

$$\begin{aligned} MK &= 2R \sin \theta \{\cos \theta + \cos 3\theta \dots \cos (2r-3)\theta\} + s \cos (2r-1)\theta \\ &= R \sin 2(r-1)\theta + s \cos (2r-1)\theta. \end{aligned}$$

Now  $GD$  is the value of  $MK$  when  $r = (k+1)$  and  $s = 0$ ,

$$\text{and so} \quad GD = R \sin 2k\theta.$$

Hence

$$MK - GD = R\{\sin 2(r-1)\theta - \sin 2k\theta\} + s \cos (2r-1)\theta.$$

Then, when  $K$  lies between  $O$  and the point of application of  $Q$

$$\begin{aligned} M_K &= M_0 + H[R\{1 + \cos 2(r-1)\theta\} - s \sin (2r-1)\theta] \\ &\quad - Q[R\{\sin 2(r-1)\theta - \sin 2k\theta\} + s \cos (2r-1)\theta] \end{aligned}$$

and when  $K$  lies between  $Q$  and  $Y$ ,

$$M_K = M_0 + H[R\{1 + \cos 2(r-1)\theta\} - s \sin (2r-1)\theta].$$

$$\text{Now} \quad \frac{\partial M_K}{\partial M_0} = 1,$$

$$\text{and} \quad \frac{\partial M_K}{\partial H} = R\{1 + \cos 2(r-1)\theta\} - s \sin (2r-1)\theta.$$

Hence on the  $r$ th side when  $K$  lies between  $O$  and  $Q$ ,

$$\begin{aligned} \left(\frac{\partial U}{\partial M_0}\right)_r &= \frac{M_0}{EI} \int_0^{2R \sin \theta} ds + \frac{H}{EI} \int_0^{2R \sin \theta} [R\{1 + \cos 2(r-1)\theta\} - s \sin (2r-1)\theta] ds \\ &\quad - \frac{Q}{EI} \int_0^{2R \sin \theta} [R\{\sin 2(r-1)\theta - \sin 2k\theta\} + s \cos (2r-1)\theta] ds, \end{aligned}$$

and when  $K$  lies between  $Q$  and  $Y$ ,

$$\left(\frac{\partial U}{\partial M_0}\right)_r = \frac{M_0}{EI} \int_0^{2R \sin \theta} ds + \frac{H}{EI} \int_0^{2R \sin \theta} [R\{1 + \cos 2(r-1)\theta\} - s \sin (2r-1)\theta] ds.$$

Upon integration this gives for K between O and Q,

$$\left(\frac{\partial U}{\partial M_0}\right)_r = \left(\frac{2R \sin \theta}{EI}\right)M_0 + \left(\frac{2R^2 \sin \theta}{EI}\right)H \left[ \frac{1 + \cos 2(r-1)\theta}{-\sin \theta \sin (2r-1)\theta} \right] \\ - \left(\frac{2R^2 \sin \theta}{EI}\right)Q \left[ \sin 2(r-1)\theta - \sin 2k\theta + \sin \theta \cos (2r-1)\theta \right].$$

When K lies between Q and Y the term involving Q is absent. The total value of  $\frac{\partial U}{\partial M_0}$  for the half structure is

$$\frac{\partial U}{\partial M_0} = \sum_{r=1}^{r=N} \left(\frac{\partial U}{\partial M_0}\right)_r.$$

Making this summation and equating to zero, we get

$$nM_0 + nRH + QR \left\{ \left(\frac{2k-1}{2}\right) \sin 2k\theta - \operatorname{cosec} \theta \sin (k-1)\theta \sin k\theta \right\} = 0. \quad (30)$$

Again, the value of  $\frac{\partial U}{\partial H}$  on the  $r$ th side when K lies between O and Q is given by

$$\left(\frac{\partial U}{\partial H}\right)_r = \frac{M_0}{EI} \int_0^{2R \sin \theta} [R\{1 + \cos 2(r-1)\theta\} - s \sin (2r-1)\theta] ds \\ + \frac{H}{EI} \int_0^{2R \sin \theta} [R\{1 + \cos 2(r-1)\theta\} - s \sin (2r-1)\theta]^2 ds \\ - \frac{Q}{EI} \int_0^{2R \sin \theta} [R\{\sin 2(r-1)\theta - \sin 2k\theta\} + s \cos (2r-1)\theta] \\ \times [R\{1 + \cos 2(r-1)\theta\} - s \sin (2r-1)\theta] ds.$$

When K lies between Q and Y the Q term is absent. Integrating and summing as before and equating to zero we obtain

$$2nM_0 + \frac{nRH}{3} \{\cos 2\theta + 8\} \\ + 2QR \left\{ \left(\frac{2k-1}{2}\right) \sin 2k\theta - \operatorname{cosec} \theta \sin (k-1)\theta \sin k\theta \right. \\ \left. - \frac{1}{12} \operatorname{cosec} 2\theta (\cos 2\theta + 2)(\cos 4k\theta - 1) \right\} = 0. \quad (31)$$

The solution of equations (30) and (31) is

$$H=Q\left\{\frac{\cos 4k\theta-1}{2n \sin 2\theta}\right\} \quad \dots \quad (32)$$

$$M_0=QR\left\{\frac{\sin k\theta\{(2k+1) \sin (k-1)\theta-(2k-1) \sin (k+1)\theta\}}{2n \sin \theta}\right\}-HR \quad (33)$$

giving the values of  $H$  and  $M_0$  for a load  $Q$  at the end of the  $k$ th side on each half of the structure.

If in the above equations we put  $\frac{k}{n}=a$  and make  $n$  infinite, we obtain the solution for a loaded circular ring

$$H=Q\left\{\frac{\cos 2a\pi-1}{2\pi}\right\} \quad \dots \quad (34)$$

$$M_0=QR\left\{\frac{1-\cos a\pi}{\pi}-a \sin a\pi+\frac{1-\cos 2a\pi}{2\pi}\right\} \quad \dots \quad (35)$$

**Case II.—Polygon with an odd number of sides.**—This case is shown at (b) in Figs. 50 and 51. Let the polygon have  $(2n+1)$  sides and let  $\theta=\frac{\pi}{2n+1}$ .

$$\begin{aligned} \text{Then} \quad MY &= R(1+\cos \theta)-OM \\ &= R\{\cos \theta+\cos 2(r-1)\theta\}-s \sin (2r-1)\theta. \end{aligned}$$

$$\begin{aligned} \text{Also} \quad KM &= R \sin 2(r-1)\theta+s \cos (2r-1)\theta \\ \text{and} \quad GD &= R \sin 2k\theta. \end{aligned}$$

Hence

$$KM-GD=R\{\sin 2(r-1)\theta-\sin 2k\theta\}+s \cos (2r-1)\theta$$

and

$$\begin{aligned} M_K &= M_0+H[R\{\cos \theta+\cos 2(r-1)\theta\}-s \sin (2r-1)\theta] \\ &\quad -Q[R\{\sin 2(r-1)\theta-\sin 2k\theta\}+s \cos (2r-1)\theta]. \end{aligned}$$

$$\text{Also} \quad \frac{\partial M_K}{\partial M_0}=1$$

$$\text{and} \quad \frac{\partial M_K}{\partial H}=R\{\cos \theta+\cos 2(r-1)\theta\}-s \sin (2r-1)\theta.$$

Hence on the  $r$ th side when  $K$  lies between  $O$  and  $Q$ ,

$$\begin{aligned} \left(\frac{\partial U}{\partial M_0}\right)_r &= \frac{1}{EI} \int_0^{2R \sin \theta} M_0 ds + \frac{H}{EI} \int_0^{2R \sin \theta} [R\{\cos \theta+\cos 2(r-1)\theta\} \\ &\quad -s \sin (2r-1)\theta] ds \\ &\quad - \frac{Q}{EI} \int_0^{2R \sin \theta} [R\{\sin 2(r-1)\theta-\sin 2k\theta\} \\ &\quad +s \cos (2r-1)\theta] ds. \end{aligned}$$

When K lies between Q and Y the Q term is absent. Then

$$\frac{\partial U}{\partial M_0} = \sum_{r=1}^{r=n} \left( \frac{\partial U}{\partial M_0} \right)_r + \frac{1}{EI} \int_0^{R \sin \theta} M_0 ds,$$

the extra integral being the value of  $\frac{\partial U}{\partial M_0}$  for the section of the frame FY.

Integrating and summing we get

$$\left( \frac{2n+1}{2} \right) M_0 + HR \left( \frac{2n+1}{2} \right) \cos \theta + QR \left\{ \left( \frac{2k-1}{2} \right) \sin 2k\theta - \operatorname{cosec} \theta \sin (k-1)\theta \sin k\theta \right\} = 0. \quad (36)$$

Again, for the region between O and Q,

$$\begin{aligned} \left( \frac{\partial U}{\partial H} \right)_r &= \frac{M_0}{EI} \int_0^{2R \sin \theta} [R\{\cos \theta + \cos 2(r-1)\theta\} - s \sin (2r-1)\theta] ds \\ &+ \frac{H}{EI} \int_0^{2R \sin \theta} [R\{\cos \theta + \cos 2(r-1)\theta\} - s \sin (2r-1)\theta]^2 ds \\ &- \frac{Q}{EI} \int_0^{2R \sin \theta} [R\{\sin 2(r-1)\theta - \sin 2k\theta\} + s \cos (2r-1)\theta] \\ &\quad \times [R\{\cos \theta + \cos 2(r-1)\theta\} - s \sin (2r-1)\theta] ds. \end{aligned}$$

Between Q and Y the Q term is absent. Then

$$\frac{\partial U}{\partial H} = \sum_{r=1}^{r=n} \left( \frac{\partial U}{\partial H} \right)_r + \frac{M_0}{EI} \int_0^{R \sin \theta} [R\{\cos \theta + \cos 2(r-1)\theta\} - s \sin (2r-1)\theta] ds,$$

the additional integral again being for the section FY.

Integrating and summing and equating to zero, we obtain the equation

$$\begin{aligned} M_0 \left( \frac{2n+1}{2} \right) \cos \theta + \frac{RH}{3} \left( \frac{2n+1}{2} \right) \left\{ 2 \cos 2\theta + \frac{5}{2} \right\} \\ + QR \left[ \cos \theta \left\{ \left( \frac{2k-1}{2} \right) \sin 2k\theta - \operatorname{cosec} \theta \sin (k-1)\theta \sin k\theta \right\} \right. \\ \left. - \frac{1}{12} \operatorname{cosec} 2\theta (\cos 2\theta + 2)(\cos 4k\theta - 1) \right] = 0. \quad (37) \end{aligned}$$

The solution of (36) and (37) is

$$H=Q\left\{\frac{\cos 4k\theta-1}{(2n+1)\sin 2\theta}\right\} \dots \dots \dots (38)$$

$$M_0=QR\left\{\frac{\sin k\theta\{(2k+1)\sin (k-1)\theta-(2k-1)\sin (k+1)\theta\}}{(2n+1)\sin \theta}\right\} \\ -HR\cos \theta \dots \dots \dots (39)$$

Putting  $\frac{2k}{(2n+1)}=a$  and making  $n$  infinite, we get the case of the loaded circular ring. This gives

$$H=Q\left\{\frac{\cos 2a\pi-1}{2\pi}\right\} \dots \dots \dots (40)$$

$$M_0=QR\left\{\frac{1-\cos a\pi}{\pi}-a\sin a\pi+\frac{1-\cos 2a\pi}{2\pi}\right\} \dots \dots (41)$$

as in equations (34) and (35).

The expression for  $H$  given by equations (32) and (38) is the same, whether the number of sides in the polygon is even or odd. The expressions for  $M_0$  differ only by the introduction of  $\cos \theta$  in one term for the case when the number of sides is odd. These equations allow the values of  $M_0$  and  $H$  to be calculated for any polygon having a load applied at the end of the  $k$ th side.

If there are more than one pair of loads, the values of  $M_0$  and  $H$  are calculated separately for each pair and then added algebraically. The sums thus obtained are the values required under the action of the complete load system.

The bending moment at any section in the framework is then readily obtained and so the bending stresses.

The vertical component of the resultant force at the section is obtained from considerations of the equilibrium of the section under the action of the external loads; the horizontal component is equal to  $H$ .

Hence the forces normal and tangential to the section can be found and the stresses due to them can be calculated. These, added to the bending stresses already found, give the total stresses at the section.

## SECTION II.—LOADS APPLIED IN PLANE OF FRAME PERPENDICULAR TO ITS AXIS OF SYMMETRY.

The arrangement of loading is shown in Fig. 52. As before one pair of loads only is taken into account, as the principle of

superposition allows the effects of any number of pairs acting together to be obtained from the results for a single pair.

The procedure is actually the same as for the case of Section I, and need not be detailed. The results are as follows :—

For a polygon with an even number of sides :

$$H=Q\left\{\frac{2k \sin 2\theta - \sin 4k\theta}{2n \sin 2\theta}\right\} \cdot \cdot \cdot \cdot (42)$$

$$M_0=QR\left\{\frac{\sin 2k\theta \cos \theta - 2k \cos 2k\theta \sin \theta}{2n \sin \theta}\right\} - HR. (43)$$

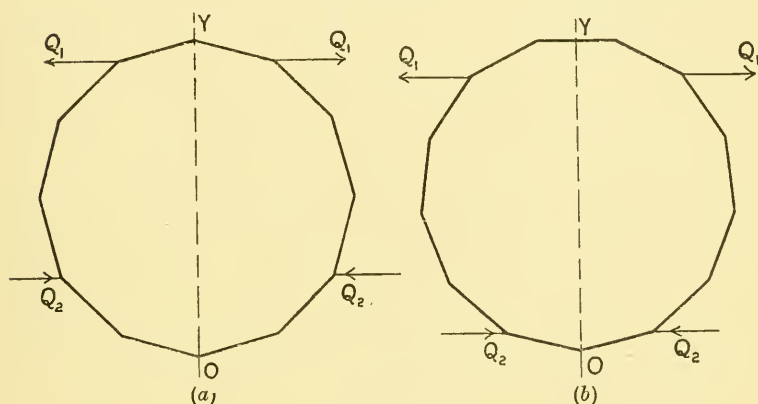


FIG. 52.

For a polygon with an odd number of sides :

$$H=Q\left\{\frac{2k \sin 2\theta - \sin 4k\theta}{(2n+1) \sin 2\theta}\right\} \cdot \cdot \cdot \cdot (44)$$

$$M_0=QR\left\{\frac{\sin 2k\theta \cos \theta - 2k \cos 2k\theta \sin \theta}{(2n+1) \sin \theta}\right\} - HR \cos \theta. (45)$$

### SECTION III.—LOADS APPLIED PERPENDICULAR TO THE PLANE OF THE FRAME.

The loads, applied to the nodes of the frame, are assumed to be symmetrically disposed about an axis of symmetry in the plane of the frame, and these loads form a system in equilibrium. Any system of this sort can be split up into a number of simple systems consisting of two pairs of loads such as  $Q_1$  and  $Q_2$  and a load  $Q_3$  at the origin  $O$  (Fig. 53), under the action of each of which the frame is in equilibrium. The effect of such a simple system will be investigated.

If the ring be imagined to be cut through at the point Y, considerations of symmetry show that the only necessary restraint

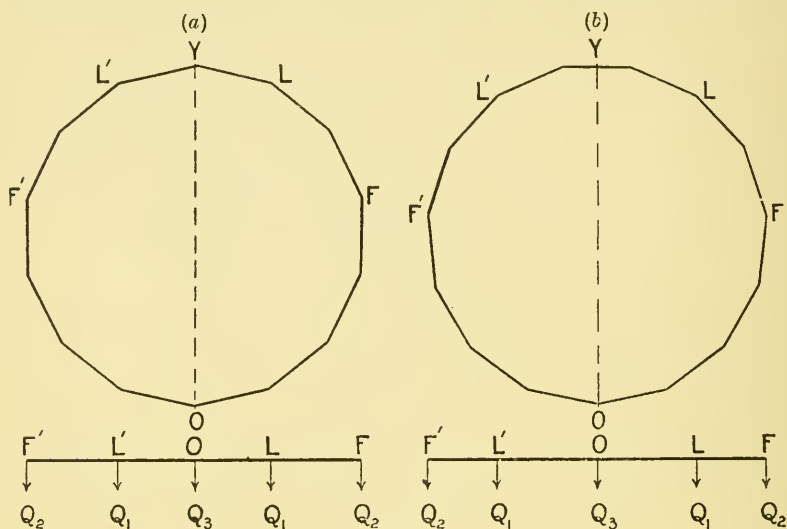


FIG. 53.

required to maintain the original condition of the ring, is a couple  $M_0$ , with axis OY, acting on each section as shown in Fig. 54.

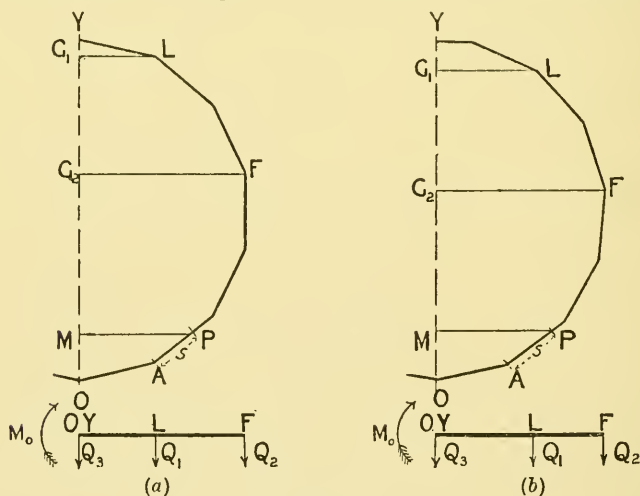


FIG. 54.

If  $U$  is the total strain energy of half the ring, then since there is no angular rotation of the section Y, we can write

$$\frac{\partial U}{\partial M_0} = 0.$$

The strain energy due to bending and torque will be alone considered.

Then if  $M$  is the bending moment and  $T$  is the torque at any point,

$$U = \frac{1}{2} \int \frac{M^2}{EI} ds + \frac{1}{2} \int \frac{T^2}{NJ} ds,$$

where  $EI$  and  $NJ$  are the flexural and torsional rigidities respectively.

Then, since these quantities are constant for the system,

$$\frac{\partial U}{\partial M_0} = \frac{1}{EI} \int M \frac{\partial M}{\partial M_0} ds + \frac{1}{NJ} \int T \frac{\partial T}{\partial M_0} ds.$$

In Fig. 54, the point  $O$  is taken as origin and loads  $Q_1$ ,  $Q_2$ , and  $Q_3$  are applied at the ends of the  $k_1$ th and  $k_2$ th sides from the origin, and at  $O$ .

The convention of sign will be clear from Fig. 54, where  $M_0$  is shown as a positive moment.

**Case I.—Polygon with an even number of sides.**

Let the polygon have  $2n$  sides, Fig. 53 (*a*) and Fig. 54 (*a*), and let the angle subtended at the centre by a side be  $2\theta$ .

Then 
$$\theta = \frac{\pi}{2n}.$$

Let  $R$  be the radius of the circumscribing circle to the polygon.

At any point  $P$ , on the  $r$ th side from the origin  $O$  and lying between  $O$  and  $F$ , the point of application of  $Q_2$ , the bending moment is given by

$$M_P = [M_0 - Q_1(MP - G_1L) - Q_2(MP - G_2F)] \cos (2r-1)\theta \\ + [Q_1(MY - G_1Y) + Q_2(MY - G_2Y)] \sin (2r-1)\theta,$$

and the torque is given by

$$T_P = [M_0 - Q_1(MP - G_1L) - Q_2(MP - G_2F)] \sin (2r-1)\theta \\ - [Q_1(MY - G_1Y) + Q_2(MY - G_2Y)] \cos (2r-1)\theta.$$

If the distance of the point  $P$  measured from the beginning of the  $r$ th side be  $s$ , then

$$MP = R \sin 2(r-1)\theta + s \cos (2r-1)\theta \\ G_1L = R \sin 2k_1\theta, \quad G_2F = R \sin 2k_2\theta$$

Hence

$$MP - G_1L = R\{\sin 2(r-1)\theta - \sin 2k_1\theta\} + s \cos (2r-1)\theta \\ MP - G_2F = R\{\sin 2(r-1)\theta - \sin 2k_2\theta\} + s \cos (2r-1)\theta.$$

Then

$$\begin{aligned} MY &= R\{1 + \cos 2(r-1)\theta\} - s \sin (2r-1)\theta \\ G_1 Y &= R\{1 + \cos 2k_1\theta\}, \quad G_2 Y = R\{1 + \cos 2k_2\theta\}. \end{aligned}$$

Hence

$$\begin{aligned} MY - G_1 Y &= R\{\cos 2(r-1)\theta - \cos 2k_1\theta\} - s \sin (2r-1)\theta \\ MY - G_2 Y &= R\{\cos 2(r-1)\theta - \cos 2k_2\theta\} - s \sin (2r-1)\theta. \end{aligned}$$

Therefore when P lies between O and the point of application of  $Q_2$ ,

$$\begin{aligned} M_P &= M_0 \cos (2r-1)\theta - Q_1[s - R\{\sin \theta + \sin (2k_1-2r+1)\theta\}] \\ &\quad - Q_2[s - R\{\sin \theta + \sin (2k_2-2r+1)\theta\}] \\ T_P &= M_0 \sin (2r-1)\theta - Q_1 R\{\cos \theta - \cos (2k_1-2r+1)\theta\} \\ &\quad - Q_2 R\{\cos \theta - \cos (2k_2-2r+1)\theta\}. \end{aligned}$$

When P lies between the points of application of  $Q_2$  and  $Q_1$ , the  $Q_2$  terms are absent, and when between the point of application of  $Q_1$  and Y

$$\begin{aligned} M_P &= M_0 \cos (2r-1)\theta \\ T_P &= M_0 \sin (2r-1)\theta. \end{aligned}$$

Now

$$\frac{\partial M_P}{\partial M_0} = \cos (2r-1)\theta \quad \text{and} \quad \frac{\partial T_P}{\partial M_0} = \sin (2r-1)\theta.$$

Hence, when P lies between O and the point of application of  $Q_2$ , the  $r$ th side contributes to  $\frac{\partial U}{\partial M_0}$  the amount

$$\begin{aligned} \left(\frac{\partial U}{\partial M_0}\right)_r &= \frac{1}{EI} \int_0^{2R \sin \theta} \left[ M_0 \cos^2 (2r-1)\theta \right. \\ &\quad - Q_1 \left\{ s \cos (2r-1)\theta \right. \\ &\quad \left. - R \cos (2r-1)\theta [\sin \theta + \sin (2k_1-2r+1)\theta] \right\} \\ &\quad \left. - Q_2 \left\{ s \cos (2r-1)\theta \right. \right. \\ &\quad \left. \left. - R \cos (2r-1)\theta [\sin \theta + \sin (2k_2-2r+1)\theta] \right\} \right] ds \\ &+ \frac{1}{NJ} \int_0^{2R \sin \theta} \left[ M_0 \sin^2 (2r-1)\theta \right. \\ &\quad - Q_1 R \sin (2r-1)\theta \{\cos \theta - \cos (2k_1-2r+1)\theta\} \\ &\quad \left. - Q_2 R \sin (2r-1)\theta \{\cos \theta - \cos (2k_2-2r+1)\theta\} \right] ds \end{aligned}$$

or

$$\begin{aligned} \left(\frac{\partial U}{\partial M_0}\right)_r = & \frac{2R \sin \theta}{EI} \left\{ \frac{M_0}{2} (1 + \cos 2(2r-1)\theta) \right. \\ & + \frac{Q_1 R}{2} (\sin 2k_1 \theta + \sin (2k_1 - 4r + 2)\theta) \\ & + \frac{Q_2 R}{2} (\sin 2k_2 \theta + \sin (2k_2 - 4r + 2)\theta) \Big\} \\ & + \frac{2R \sin \theta}{NJ} \left\{ \frac{M_0}{2} (1 - \cos 2(2r-1)\theta) \right. \\ & - \frac{Q_1 R}{2} \left( \begin{array}{l} 2 \cos \theta \sin (2r-1)\theta - \sin 2k_1 \theta \\ \quad + \sin (2k_1 - 4r + 2)\theta \end{array} \right) \\ & - \frac{Q_2 R}{2} \left( \begin{array}{l} 2 \cos \theta \sin (2r-1)\theta - \sin 2k_2 \theta \\ \quad + \sin (2k_2 - 4r + 2)\theta \end{array} \right) \Big\}. \end{aligned}$$

Summing corresponding expressions for all sides of the polygon, with omission of the  $Q_2$  terms when  $r$  is greater than  $k_2$ , and of both the  $Q_1$  and  $Q_2$  terms when  $r$  is greater than  $k_1$ , and equating to zero we have

$$\begin{aligned} \frac{1}{EI} \left[ M_0 \left( n + \frac{\cos 2n\theta \sin 2n\theta}{\sin 2\theta} \right) + Q_1 R k_1 \sin 2k_1 \theta + Q_2 R k_2 \sin 2k_2 \theta \right] \\ + \frac{1}{NJ} \left[ M_0 \left( n - \frac{\cos 2n\theta \sin 2n\theta}{\sin 2\theta} \right) \right. \\ \quad - Q_1 R \left( \frac{2 \cos \theta \sin^2 k_1 \theta}{\sin \theta} - k_1 \sin 2k_1 \theta \right) \\ \quad \left. - Q_2 R \left( \frac{2 \cos \theta \sin^2 k_2 \theta}{\sin \theta} - k_2 \sin 2k_2 \theta \right) \right] = 0, \end{aligned}$$

which may be written

$$\begin{aligned} \frac{1}{EI} [M_0 n + Q_1 R k_1 \sin 2k_1 \theta + Q_2 R k_2 \sin 2k_2 \theta] \\ + \frac{1}{NJ} [M_0 n - Q_1 R \{ \cot \theta (1 - \cos 2k_1 \theta) - k_1 \sin 2k_1 \theta \} \\ - Q_2 R \{ \cot \theta (1 - \cos 2k_2 \theta) - k_2 \sin 2k_2 \theta \}] = 0, \end{aligned}$$

giving

$$\begin{aligned} M_0 = & \frac{1}{n \left( \frac{1}{EI} + \frac{1}{NJ} \right)} \left[ Q_1 R \left\{ \frac{\cot \theta}{NJ} (1 - \cos 2k_1 \theta) - \left( \frac{1}{EI} + \frac{1}{NJ} \right) k_1 \sin 2k_1 \theta \right\} \right. \\ & \left. + Q_2 R \left\{ \frac{\cot \theta}{NJ} (1 - \cos 2k_2 \theta) - \left( \frac{1}{EI} + \frac{1}{NJ} \right) k_2 \sin 2k_2 \theta \right\} \right] \end{aligned} \quad (46)$$

Again, since the frame is in equilibrium under the load  $Q_3$  and the pairs of loads  $Q_1$  and  $Q_2$  we have the two static conditions

$$2Q_1 + 2Q_2 + Q_3 = 0 \quad . \quad . \quad . \quad . \quad (47)$$

and

$$Q_1(1 - \cos 2k_1\theta) + Q_2(1 - \cos 2k_2\theta) = 0 \quad . \quad . \quad (48)$$

and therefore

$$M_0 = \frac{R}{n} (-Q_1 k_1 \sin 2k_1\theta - Q_2 k_2 \sin 2k_2\theta) \quad . \quad (49)$$

**Case II.—Polygon with an odd number of sides.**—The same procedure is followed and leads to the same results with the substitution of  $(2n+1)$  for  $2n$ .

In either case,  $M_0$  having been calculated, the values of  $M_P$  and  $T_P$ , and therefore the stress at any point in the frame, may be found.

If the frame is in equilibrium under a symmetrical system of loads  $W_0$  at  $O$  and pairs of loads  $W_1, W_2, W_3$ , etc., at the ends of the first, second, third, etc., sides from  $O$ , then this system may be split up into groups of loads such as  $Q_1, Q_2$ , and  $Q_3$  above, under the action of each of which the frame is in equilibrium.

Therefore for a frame of  $2n$  sides

$$M_0 = -\frac{R}{n} [W_1 \sin 2\theta + 2W_2 \sin 4\theta + 3W_3 \sin 6\theta + \dots \\ + (n-1)W_{(n-1)} \sin 2(n-1)\theta] \quad . \quad (50)$$

and for a frame of  $(2n+1)$  sides

$$M_0 = -\frac{R}{\left(\frac{2n+1}{2}\right)} [W_1 \sin 2\theta + 2W_2 \sin 4\theta + 3W_3 \sin 6\theta + \dots \\ + nW_n \sin 2n\theta] \quad . \quad (51)$$

## EXAMPLE 3

### THE STRESSES AND DISTORTION OF LINKS

THIS example illustrates the graphical integration of the equations derived from strain energy conditions when these equations cannot be dealt with by mathematical means.

It also includes a number of experimental results made to check the accuracy of the method of analysis.

The original work was published by The Institution of Mechanical Engineers.<sup>1</sup>

**General Theory.**—Consider a link as shown in Fig. 55, symmetrical about the axis AB and subjected to equal and opposite pulls of magnitude  $2P$ . If this link be supposed cut at the section A, the right-hand half carries a load  $P$ , and, in order to restore the initial conditions of equilibrium, it is necessary to apply a couple  $M_0$  and a horizontal force  $H$  to the section A (Fig. 56).

The bending moment at any point X is

$$M_x = M_0 + Hy - Px \quad . \quad . \quad . \quad . \quad . \quad (52)$$

where  $x$  and  $y$  are the distances from X to the lines of action of  $P$  and  $H$  respectively.

The effect of neglecting the curvature in estimating the strain energy due to the bending moment is small and the work is considerably simplified, but the effect of curvature is important and must be considered when estimating the stresses due to bending.

In order to evaluate  $M_x$ , it is necessary to determine  $M_0$  and  $H$ , and this is done as follows :—

Let  $U$  be the total strain energy in the right-hand half of the link due to the action of  $P$ ,  $H$ , and  $M_0$ . The angular rotation of the section A is given by  $\frac{\partial U}{\partial M_0}$ , and the movement of A in the

<sup>1</sup> Inst. Mech. Engs. *Proceedings*, December, 1923. "The Stresses in Links and their Alteration in Length under Load." A. J. S. Pippard and C. V. Miller.

direction of  $H$  is given by  $\frac{\partial U}{\partial H}$ . From consideration of symmetry both these are zero, so

$$\frac{\partial U}{\partial H} = 0,$$

$$\frac{\partial U}{\partial M_0} = 0,$$

and these conditions enable the values of  $M_0$  and  $H$  to be found.

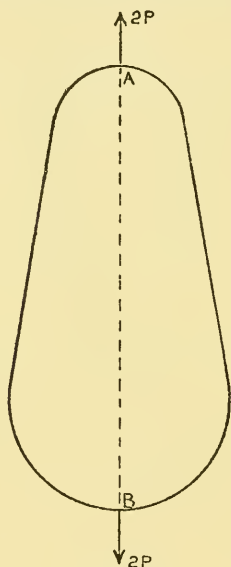


FIG. 55.

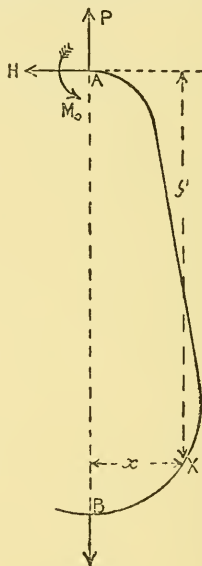


FIG. 56.

The total strain energy is the sum of three terms :—

- (a) That due to bending.
- (b) That due to tangential force.
- (c) That due to normal force.

The effect of the last two is generally negligible compared with the first, and exercises no appreciable effect on  $M_0$  and  $H$ . To show the relative importance, however, the terms due to the tangential force are kept in the equations. When the link is symmetrical about AB, and also about an axis at right angles to AB, as usually happens, the bending moment at B is

$$M_B = M_0 + H \cdot BA.$$

Since by symmetry  $M_B = M_0$ , we have  $H = 0$ , and  $M_0$  alone has to be determined.

**Strain Energy due to Bending.**—The strain energy due to bending is

$$U_B = \frac{1}{2E} \int_0^{s_A} \frac{M_x^2}{I} ds,$$

where  $s$  is the distance of the point  $X$  (Fig. 56) measured from  $B$  along the curve;  $s_A$  is the value of  $s$  at  $A$ ;  $I$  is the moment of inertia of the cross-section of the link at  $X$ , and  $E$  is the modulus of elasticity of the material. Hence

$$\frac{\partial U_B}{\partial M_0} = \frac{1}{E} \int_0^{s_A} \frac{M_x}{I} \frac{\partial M_x}{\partial M_0} ds,$$

and

$$\frac{\partial U_B}{\partial H} = \frac{1}{E} \int_0^{s_A} \frac{M_x}{I} \frac{\partial M_x}{\partial H} ds.$$

But from equation (52),

$$\frac{\partial M_x}{\partial M_0} = 1 \quad \text{and} \quad \frac{\partial M_x}{\partial H} = y.$$

Substituting these values, and the expression for  $M_x$  from equation (52) we obtain

$$\frac{\partial U_B}{\partial H} = \frac{1}{E} \left\{ M_0 \int_0^{s_A} \frac{y}{I} ds + H \int_0^{s_A} \frac{y^2}{I} ds - P \int_0^{s_A} \frac{xy}{I} ds \right\} \quad (53)$$

and

$$\frac{\partial U_B}{\partial M_0} = \frac{1}{E} \left\{ M_0 \int_0^{s_A} \frac{ds}{I} + H \int_0^{s_A} \frac{y}{I} ds - P \int_0^{s_A} \frac{x}{I} ds \right\} \quad (54)$$

**Strain Energy due to Tangential and Normal Force.**—At any point  $X$  on the link there is a vertical force  $P$  and a horizontal force  $H$ .

If the tangent to the curve at  $X$  makes an angle  $\theta$  with the inwardly directed horizontal line at  $X$ , Fig. 57, by resolution of the forces  $H$  and  $P$  along and perpendicular to the tangent, the tangential component of force at  $X$  is

$$T_x = (H \cos \theta + P \sin \theta) \quad (55)$$

and the normal component of force at X is

$$J_x = (H \sin \theta - P \cos \theta) \quad . \quad (56)$$

The strain energy due to the tangential component is

$$U_T = \frac{1}{2E} \int_0^{s_A} \frac{T^2}{A} ds \quad . \quad (57)$$

and due to the normal component

$$U_J = \frac{1}{2E_s} \int_0^{s_A} \frac{J^2}{A} ds \quad . \quad (58)$$

where A is the cross-sectional area of the link at X and  $E_s$  is the modulus of rigidity of the material. From (57) and (55)

$$\frac{\partial U_T}{\partial H} = \frac{1}{E} \int_0^{s_A} \frac{1}{A} \cos \theta (H \cos \theta + P \sin \theta) ds.$$

Also

$$\frac{\partial U_T}{\partial M_0} = 0.$$

By equating to zero the complete values for  $\frac{\partial U}{\partial H}$  and  $\frac{\partial U}{\partial M_0}$  (neglecting the effect of shear), we find

$$\begin{aligned} M_0 \int_0^{s_A} \frac{y}{I} ds + H \left( \int_0^{s_A} \frac{y^2}{I} ds + \int_0^{s_A} \frac{1}{A} \cos^2 \theta ds \right) \\ + P \left( \int_0^{s_A} \frac{1}{A} \sin \theta \cos \theta ds - \int_0^{s_A} \frac{xy}{I} ds \right) = 0 \quad . \quad (59) \end{aligned}$$

$$M_0 \int_0^{s_A} \frac{1}{I} ds + H \int_0^{s_A} \frac{y}{I} ds - P \int_0^{s_A} \frac{x}{I} ds = 0 \quad . \quad . \quad (60)$$

These simultaneous equations give  $M_0$  and H in terms of P. In general the integrals cannot be evaluated directly, and graphical methods must be used.

**Graphical Evaluation of Terms in the Equations.**—Draw the curve joining the centroids of all cross-sections of the link and

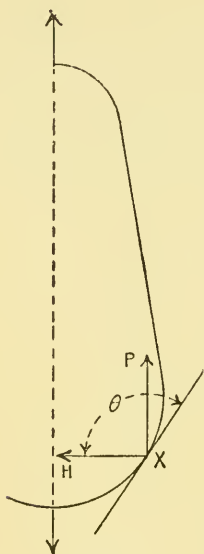


FIG. 57.

divide it into any number of equal parts. At each point measure the values of  $s$ ,  $x$ ,  $y$ , and  $\theta$ , and calculate the values of  $A$  and  $I$ . Calculate  $\frac{y}{I}$ ,  $\frac{y^2}{I}$ ,  $\frac{1}{A} \cos^2 \theta$ ,  $\frac{1}{A} \sin \theta \cos \theta$ ,  $\frac{xy}{I}$ ,  $\frac{1}{I}$ , and  $\frac{x}{I}$ , and plot each against the value of  $s$ . The areas of the curves so obtained, after correcting for scales, give the values of the integrals in equations (59) and (60). Substituting these, the equations can be solved and  $M_0$  and  $H$  found in terms of  $P$ .

The cases worked out in detail later will enable the procedure to be clearly followed.

If the link is symmetrical about two axes at right angles,  $H=0$  and equation (60) gives

$$M = \frac{\int_0^{s_A} \frac{x}{I} ds}{\int_0^{s_A} \frac{1}{I} ds} P \quad \dots \quad (61)$$

and two curves only are needed to obtain the value of  $M_0$ .

**Stresses at any Point.**—The bending moment, tangential force and shear force at any section of the link can now be found from the equations (52), (55), and (56), and the stresses determined by the usual methods.<sup>1</sup>

**Calculation of the Alteration in Length of Link.**—The values of  $M_0$  and  $H$  being functions of  $P$ , the strain energy in the link can be expressed in terms of this quantity. The movement of  $A$  relative to  $B$  in the line of  $P$  is then given by  $\frac{dU}{dP}$ , where

$$U = \frac{1}{2} \left\{ \frac{1}{E} \int_0^{s_A} \frac{M_x^2}{I} ds + \frac{1}{E} \int_0^{s_A} \frac{T_x^2}{A} ds + \frac{1}{E_s} \int_0^{s_A} \frac{J_x^2}{A} ds \right\} \quad (62)$$

If  $M_x$ ,  $T_x$ , and  $J_x$  are expressed in terms of  $P$ , and curves of  $\frac{M_x^2}{I}$ ,  $\frac{T_x^2}{A}$ , and  $\frac{J_x^2}{A}$  are plotted against  $s$ , the areas of these curves give the values of the integrals in equation (62), and thus the value of  $U$ .

Let  $A_M P^2$ ,  $A_T P^2$ ,  $A_J P^2$  represent these areas after correction for scale.

<sup>1</sup> *Strength of Materials*, Morley, chap. xii.

Then for the complete link the strain energy is

$$U = \frac{W^2}{4} \left[ \frac{A_M}{E} + \frac{A_P}{E} + \frac{A_J}{E_s} \right],$$

where  $W = 2P$ .

The movement of A relative to B in the line of W is

$$\Delta = \frac{dU}{dW} = \frac{W}{2} \left[ \frac{A_M}{E} + \frac{A_P}{E} + \frac{A_J}{E_s} \right] \quad \dots (63)$$

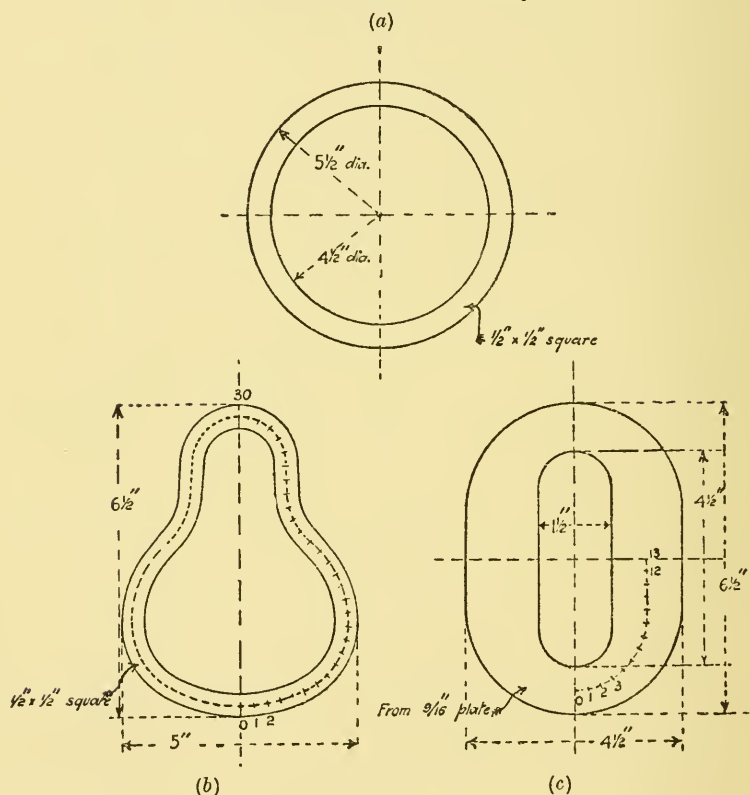


FIG. 58.

**Application of Method.**—To check the degree of accuracy of the method just described, calculations and tests were made on the three links shown in Fig. 58. In order to eliminate possible errors due to welding, these were cut out of steel plate: the modulus of elasticity of the metal was experimentally found to be  $28.5 \times 10^6$  lb. per square inch.

The object of testing the circular link (a) was to find the best

method of experimentation, as in this case the alteration of length under load can be determined analytically. The calculations were made and the link tested by a tensile loading, measurements being made of the outside diameter at different loads by means of a bar vernier. This method was found to be insufficiently accurate, and a similar link was tested under compressive loads, the shortening of the inside diameter being measured by a micrometer. This gave satisfactory and consistent results and the same method of test was adopted for the other links. The details of the calculations and the test figures for each link are given below.

**Calculations and Tests : Circular Ring, Fig. 58 (a).**—The value of  $M_0$  in this case is given by Morley <sup>1</sup> as  $\frac{WR}{\pi}$ , where W is the load and R is the mean radius.

Then if  $\theta$  be the angle between the radius to the top of the link and that to any point X on the circumference

$$M_x = WR \left( \frac{1}{\pi} - \frac{1}{2} \sin \theta \right)$$

$$T_x = \frac{W}{2} \sin \theta$$

$$J_x = \frac{W}{2} \cos \theta,$$

and the deflections due to these separate terms are found to be

$$\Delta_B = 0.00001568W,$$

$$\Delta_T = 0.000000275W,$$

$$\Delta_J = 0.000000551W,$$

giving a total deflection

$$\Delta = 0.00001651W \quad . \quad . \quad . \quad . \quad . \quad (64)$$

where  $\Delta$  is in inches and W in lb.

The test figures for this ring under compression are given in Table I, together with the calculated values.

It will be noticed that the values calculated from equation (64) differ from the actual test figures only in the fourth decimal place, except in one instance, which is probably due to an incorrect reading of the micrometer. The micrometer was only graduated to 0.001 inch and the agreement between test and calculation may be considered satisfactory.

**Special Link, Fig. 58 (b).**—The calculations for a few points on

<sup>1</sup> *Strength of Materials*, 1st ed., Section 133.

this link are given in Table II. Since I and A are constants, they are not introduced at this stage of the work.

TABLE I.—CIRCULAR RING UNDER COMPRESSION.

Load W.		Internal diameter.	Alteration in diameter.		
Actual.	Increment.		Measured.	Calculated.	Error.
Ton.	Ton.	Inches.	Inch.	Inch.	Inch.
0.15	—	4.4985	—	—	—
0.20	0.05	4.4965	0.0020	0.00185	0.00015
0.25	0.10	4.4958 <sup>1</sup>	0.0027	0.00370	0.00100
0.30	0.15	4.4930	0.0055	0.00555	0.00005
0.35	0.20	4.4915	0.0070	0.00740	0.00040
0.40	0.25	4.4890	0.0095	0.00925	0.00025
0.45	0.30	4.4880	0.0105	0.01110	0.00050
0.50	0.35	4.4860	0.0125	0.01295	0.00045
0.55	0.40	4.4840	0.0145	0.01480	0.00030
0.60	0.45	4.4820	0.0165	0.01665	0.00015
0.65	0.50	4.4710	0.0275	0.01850	Yielded.

<sup>1</sup> Probably a misreading of the micrometer for 4.4948.

TABLE II.

Point.	<i>x</i>	<i>y</i>	<i>xy</i>	<i>y</i> <sup>2</sup>	$\theta$	$\sin \theta$	$\cos \theta$	$\sin \theta \cos \theta$	$\cos^2 \theta$
	Ins.	Ins.	Ins. <sup>2</sup>	Ins. <sup>2</sup>	Deg				
0	0	6.00	0	36.0	180.0	0	—1.000	0	1.0
3	0.82	5.89	4.83	34.7	161.2	0.272	—0.962	—0.262	0.925
6	1.58	5.54	8.75	30.7	144.2	0.585	—0.811	—0.474	0.658
9	2.11	4.91	10.36	24.1	114.8	0.908	—0.419	—0.381	0.176
12	2.24	4.11	9.21	16.9	86.3	0.998	0.064	0.064	0.004
15	2.03	3.30	6.70	10.9	63.4	0.890	0.448	0.397	0.201
18	1.53	2.65	4.06	7.02	44.1	0.696	0.718	0.500	0.515
21	1.09	1.95	2.13	3.81	74.8	0.965	0.262	0.253	0.069
24	0.98	1.13	1.11	1.28	83.4	0.990	0.115	0.114	0.013
27	0.74	0.35	0.26	0.12	49.8	0.764	0.645	0.493	0.416
30	0	0	0	0	0	0	1.000	0	1.000

From the complete calculations the necessary curves were plotted (Fig. 59), and their areas corrected for scales were :

$$\begin{aligned}
 \int ds &= 8.4 \text{ inches.} & \int xy ds &= 40.25 \text{ inches}^3. \\
 \int y ds &= 27.72 \text{ inches}^2. & \int \sin \theta \cos \theta ds &= 0.58 \text{ inch.} \\
 \int x ds &= 11.3 \text{ inches}^2. & \int \cos^2 \theta ds &= 3.3 \text{ inches.} \\
 \int y^2 ds &= 124.25 \text{ inches}^3.
 \end{aligned}$$

The cross-section of the link was  $\frac{1}{2}$  inch square throughout. Hence  $I = \frac{1}{192}$  inch units and  $A = 0.25$  square inch are constant throughout the link.

Substituting the values of the integrals and the values of  $A$ ,  $I$  and  $E$  in equations (59) and (60), we obtain

$$\begin{aligned} 5320M_0 + H(23850 + 14) + P(1 - 7930) &= 0, \\ 1611M_0 + 5320H - 2165P &= 0. \end{aligned}$$

The terms in the first of these equations due to direct stress, namely,  $14H$  and  $P$ , are negligible compared with those due to bending. Their omission would have resulted in a great saving of work, since no trigonometrical functions would have occurred in the calculations.

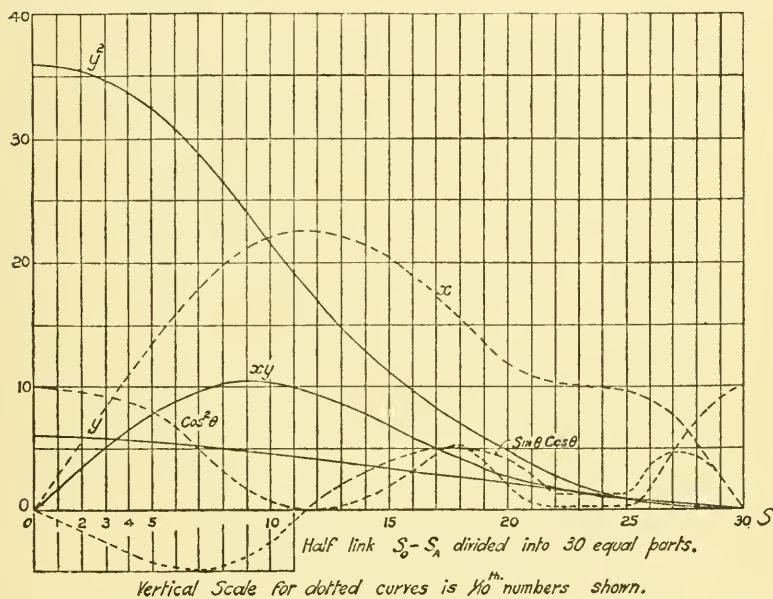


FIG. 59.

From these equations we obtain

$$\begin{aligned} H &= 0.126P \text{ lb.} \\ M_0 &= 0.925P \text{ inch-lb.} \end{aligned}$$

Hence from equations (52), (55) and (56)

$$\begin{aligned} M_x &= P(0.925 + 0.126y - x), \\ T_x &= P(0.126 \cos \theta + \sin \theta), \\ J_x &= P(0.126 \sin \theta - \cos \theta). \end{aligned}$$

The values of  $M_x$ ,  $T_x$  and  $J_x$  can now be determined for all points in terms of  $P$ , and the strain energy calculated.

Table III gives calculations for certain points on the link.

TABLE III.

Point.	$0.126y$	$x$	$0.126y - x$	$\frac{M}{P}$	$\frac{M^2}{P^2}$	$0.126 \cos \theta$	$0.126 \cos \theta + \sin \theta$	$\frac{T^2}{P^2}$	$0.126 \sin \theta$	$0.126 \sin \theta - \cos \theta$	$\frac{J^2}{P^2}$
	Inch.	Inch.	Inch.	Inch.	Inch. <sup>2</sup>						
0	0.756	—	0.756	1.681	2.840	-0.126	-0.126	0.016	—	1.0	1.0
3	0.742	0.83	-0.088	0.837	0.700	-0.1210	0.155	0.024	0.0344	0.996	0.992
6	0.698	1.58	-0.882	0.043	0.002	-0.1020	0.484	0.234	0.0737	0.885	0.782
9	0.619	2.10	-1.481	-0.556	0.310	-0.0525	0.857	0.739	0.1145	0.533	0.284
12	0.516	2.24	-1.724	-0.799	0.640	0.0079	1.006	1.012	0.1257	0.062	0.004
15	0.416	2.02	-1.604	-0.679	0.460	0.0565	0.950	0.905	0.1120	-0.336	0.113
18	0.334	1.52	-1.186	-0.261	0.068	0.0905	0.786	0.619	0.0876	-0.630	0.398
21	0.246	1.09	-0.844	0.081	0.006	0.0330	0.998	0.999	0.1215	-0.140	0.020
24	0.142	0.980	-0.838	0.087	0.008	0.0145	1.007	1.014	0.1260	0.011	—
27	0.044	0.74	-0.696	0.229	0.052	0.0810	0.847	0.720	0.0962	-0.5488	0.301
30	—	—	—	0.925	0.851	0.1260	0.126	0.016	—	-1.000	1.000

The areas of the curves obtained from the complete calculation give the following values of the integrals :—

$$\int M^2 ds = 3.25P^2 \text{ (inch-lb. units).}$$

$$\int T^2 ds = 5.26P^2 \quad \text{,,} \quad \text{,,}$$

$$\int J^2 ds = 3.22P^2 \quad \text{,,} \quad \text{,,}$$

From these figures the total strain energy of the link is

$$U = 0.000005875W^2 \text{ inch-lb.}$$

where  $W = 2P$ ,

and  $\Delta = \frac{dU}{dW} = 0.00001175W \text{ inches.}$

The test figures for this link in compression, together with the calculated values, are given in Table IV.

TABLE IV.

Load.		Diameter.	Alteration in length.		
Actual.	Increment.		Measured.	Calculated.	Error.
Ton.	Ton.	Inches.	Inch.	Inch.	Inch.
0.15	—	4.4928	—	—	—
0.25	0.10	4.4903	0.0025	0.00264	0.0001
0.30	0.15	4.4890	0.0038	0.00396	0.00016
0.35	0.20	4.4875	0.0053	0.00528	0.00002
0.40	0.25	4.4850	0.0078	0.00660	0.0012
0.45	0.30	4.4845	0.0083	0.0079	0.0004
0.50	0.35	4.4830	0.0098	0.0092	0.0006
0.55	0.40	4.4800	Yield	—	—

With the exception of one point the whole of the readings are within the allowable limits of accuracy.

**Special Link, Fig. 58 (c).**—This link more nearly represents a practical form ; it is symmetrical about a horizontal as well as a vertical line, and has a varying cross-section. Equation (61) may be used to determine  $M_0$ , and one quadrant only need be considered when plotting the curves. The calculations for selected points are given in Table V, and the curves of  $\frac{x}{l}$  and  $\frac{1}{l}$  plotted against  $s$  are shown in Fig. 60.

TABLE V.  
(Units are inch, degree and lb.)

Point.	$x$	Width of section.	$\theta$	$\sin \theta$	$\cos \theta$	$\frac{1}{I}$	$\frac{x}{I}$	$\frac{M}{P}$	$\frac{M^2}{IP^2}$	A	$\frac{\sin^2 \theta}{A}$	$\frac{\cos^2 \theta}{A}$
0	0	1.000	180	0	-1.000	21.33	0	1.074	24.6	0.562	0	1.78
3	0.789	1.087	148	0.5299	-0.8480	16.67	13.15	0.285	1.35	0.612	0.460	1.18
6	1.336	1.302	118	0.8829	-0.4695	9.66	12.90	-0.262	0.66	0.732	1.068	0.30
9	1.496	1.493	90	1.000	0	6.40	9.58	-0.422	1.14	0.840	1.190	0
13	1.496	1.493	90	1.000	0	6.40	9.58	-0.422	1.14	0.840	1.190	0

The areas of these curves are

$$\int_0^x \frac{x}{I} ds \quad \text{for one quadrant} = 8.1 \text{ square inches.}$$

$$\int_0^1 \frac{1}{I} ds \quad \text{for one quadrant} = 7.55 \text{ square inches.}$$

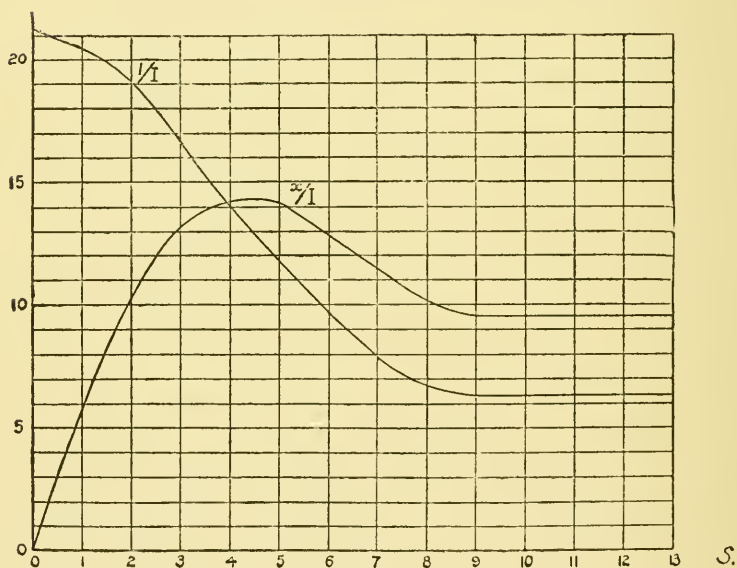


FIG. 60.

The curves were plotted to the same scale, and since the value of  $M_0$  is the ratio of these integrals, no scale corrections were required. Hence from equation (61),  $M_0 = 1.074P$  inch lb.

Then

$$M_x = P(1.074 - x),$$

$$T = P \sin \theta,$$

$$J = -P \cos \theta.$$

Since  $I$  and  $A$  are not constant, it is necessary, in order to estimate the alteration in length of the link, to plot the values of  $\frac{M_x^2}{I}$ ,  $\frac{T^2}{A}$ , and  $\frac{J^2}{A}$ . Fig. 61 shows these curves for one quadrant, and the values of the integrals for the complete link are

$$\begin{aligned}\int \frac{M_x^2}{IP^2} ds &= 46, \\ \int \frac{T^2}{AP^2} ds &= 12.46, \\ \int \frac{J^2}{AP^2} ds &= 7.84.\end{aligned}$$

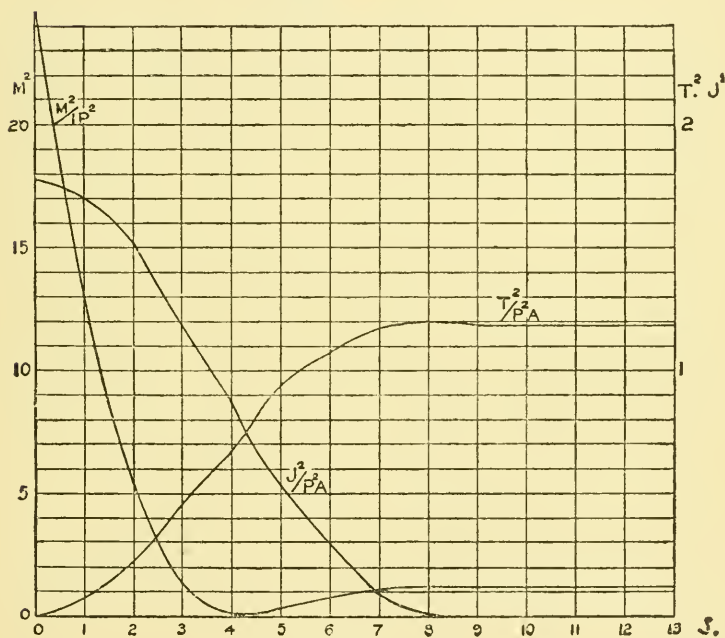


FIG. 61.

If we put  $W=2P$ , the total strain energy is

$$U=0.000000327W^2,$$

and the alteration in length under a load  $W$  is

$$\Delta = \frac{dU}{dW} = 0.000000654W.$$

The test figures obtained for this link are given in Table VI.

The error between observed and calculated values is in no place greater than 0.0005 inch, which in view of the method of measurement is very satisfactory agreement.

TABLE VI.

Load W.	Load increment.	Diameter.	Alteration in length.		
			Measured.	Calculated.	Error.
Tons.	Tons.	Inches.	Inch.	Inch.	Inch.
0.5	—	4.5025	—	—	—
1.0	0.5	4.5019	0.0006	0.0007	0.0001
1.5	1.0	4.5011	0.0014	0.0015	0.0001
2.0	1.5	4.5003	0.0022	0.0022	0.0000
2.5	2.0	4.4997	0.0028	0.0029	0.0001
3.0	2.5	4.4991	0.0034	0.0037	0.0003
3.5	3.0	4.4984	0.0041	0.0044	0.0003
4.0	3.5	4.4979	0.0046	0.0051	0.0005
4.5	4.0	4.4970	0.0055	0.0059	0.0004
5.0	4.5	Yield	—	—	—

## EXAMPLE 4

### THE STRESSES IN A ROOF TRUSS HAVING CURVED BRACING MEMBERS

THIS example illustrates the method of dealing with a redundant structure when some of the members are curved and in consequence have a lower effective modulus of elasticity than the material of which they are made. The case selected is that of the roof of Westminster Hall—a wonderful example of mediæval construction. This roof was repaired by the Office of Works when it was found that extensive damage had been done to it by the death-watch beetle.

It should be pointed out that the application of strain-energy methods to timber structures is not strictly defensible as the material is very variable in properties and is not isotropic. It can, however, be justified by the fact that any analysis must assume certain properties of the material, and the present treatment is likely to be as accurate as can be expected from any theoretical method.

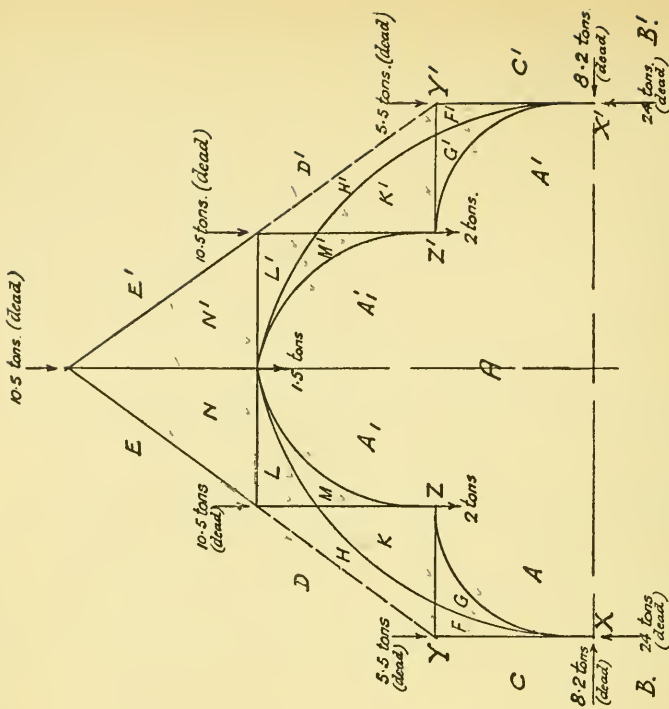
The question of accuracy of assumptions in a particular case does not, however, affect the accuracy of method, and this example may be viewed, if desired, merely as an illustration of method. It was first published by the Building Research Station.<sup>1</sup>

A drawing of the roof truss of Westminster Hall is shown in Fig. 62, together with a simplified line diagram required for the purposes of stress analysis. In order to make the problem manageable, certain assumptions have been made as follows :

- (1) The truss is assumed to be carried on the walls of the building by attaching the feet of the wall posts to pins on the walls. Thus the reactions are taken at two points only.
- (2) The walls are considered to be perfectly rigid.

<sup>1</sup> Building Research Technical Paper No. 2. "Primary Stress Determination in Timber Roofs with Special Reference to Curved Bracing Members." A. J. S. Pippard and W. H. Glanville. H.M. Stationery Office, 1926.

WESTMINSTER HALL ROOF  
(ARCHED HAMMER-BEAM TYPE)

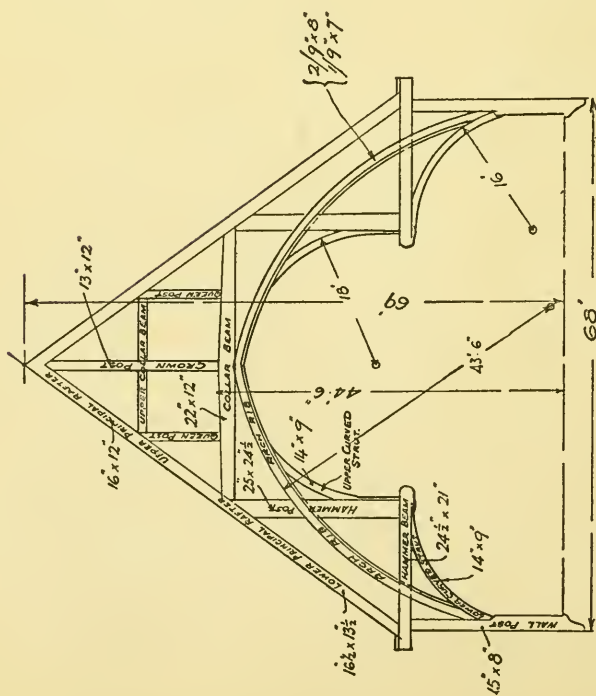


## Outline Diagram of Truss.

### Simplified for Analysis.

(Loading approximate only. Dotted members taken as redundant.)

Fig. 62.



### Elevation of Truss

- (3) The top collar and two queen posts which do not form integral portions of the trussing system are supposed to be absent.
- (4) The curved struts are supposed to be continuous between nodes of the frame as indicated on the line diagram.
- (5) The centre lines of the principal rafter, hammer beam, and wall post are supposed to meet at a point.

It will be seen from the line diagram that there are twelve nodes in the frame to be connected to the two pins X and X' on the wall, and in order that a just-stiff frame shall result, twenty-four members are required. Actually twenty-seven are provided, so that the frame as simplified has three redundancies.

As long as the essential bars form a just-stiff frame the selection of redundancies is merely a matter of convenience, and in the present case we shall take HD and H'D' as two and the horizontal thrust at the wall as the third. This symmetrical choice somewhat simplifies the calculations.

The first step is to reduce the curved bars to their equivalent straight members as explained in Part I, p. 54.

The formula to be used is

$$\frac{E}{E'} = 1 + N \left( \frac{R}{k} \right)^2,$$

and the values of N can be read from Fig. 34.

Particulars of these members and the values of  $\frac{E}{E'}$  are given in Table I.

TABLE I.

Member.	$\phi^\circ$ Half angle subtended by curved member.	N From curve, Fig. 34.	R Radius of member, ins.	$k^2$ (Radius of gyration) <sup>2</sup>	$\frac{E}{E'} = 1 + N \left( \frac{R}{k} \right)^2$
AG	45.0	0.0500	240	16.4	176.5
GF	13.4	0.00040	522	30.95	4.52
AM	45.0	0.0500	264	16.4	213.4
KH	9.5	0.00011	522	30.95	1.97
ML	16.15	0.00087	522	30.95	8.66

Since the exact value of E is unimportant and does not appear in the final result, it has been assumed to be  $1 \times 10^6$  lbs. per square inch, and the values of E' based on this figure are given in col. 4 of Table II.

TABLE II.—WESTMINSTER HALL ROOF.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	
Member.	Length L, ins.	Area A, ins. <sup>2</sup>	E or equiv. lb./in. <sup>2</sup>	$\lambda = \frac{L}{EA} \times 10^7$	Coefficients.			$\frac{\partial U}{\partial R_1} = \lambda P_0 \frac{\partial P_0}{\partial R_1}$			$\frac{\partial U}{\partial R_2} = \lambda P_0 \frac{\partial P_0}{\partial R_2}$			Loads in members, tons.				Stresses in members, lb. sq. in.			
					$\alpha$	$\beta$	$\gamma$	$\lambda \alpha^2$	$\lambda \alpha \beta$	$\lambda \alpha \gamma$	$\lambda \alpha \beta$	$\lambda \beta^2$	$\lambda \beta \gamma$	$\alpha R_1$	$\beta R_2$	P	P <sub>0</sub>	Direct.	Bend- ing.	Total.	
AG	310	126.0	5.66	4,348.0	-2.285	+0.480	+10.92	22,702	-4,769	-108,492		1,002	+22,790	-22.43	+3.93	+10.92	-7.58	-135	4,000	-4,135	
GF	247	252.0	221.0	44.4	+1.822	-1.230	-27.76	147	-99	-2,246		67	+1,516	+17.90	-10.70	-27.76	-20.56	-183	825	-1,008	
FC	240	120.0	1,000.0	20.0	—	+0.822	-5.50	—	—	—		14	-90	—	+6.73	-5.50	+1.23	+23	—	+23	
AM	354	126.0	4.69	5,990.0	-4.185	+0.718	+38.05	104,910	-17,999	-953,843		3,088	+163,646	-41.10	+5.88	+38.05	+2.88	+50	1,645	+1,695	
MK	152	612.5	1,000.0	2.48	+1.610	-0.208	-20.32	6	-1	-81		—	+10	+15.81	-1.70	-20.32	-6.21	-23	—	-23	
KG	136	514.5	1,000.0	2.64	-1.025	+0.119	+15.52	3	—	-42	As	—	+5	-10.07	+0.97	+15.52	+6.42	+28	—	+28	
KH	214	252.0	508.0	16.72	+2.302	-1.552	-35.05	89	-60	-1,349	Column	40	+910	+22.61	-12.71	-35.05	-25.15	-224	509	-733	
HF	60	514.5	1,000.0	1.17	—	-0.571	—	—	—	—		—	—	—	-4.68	—	-4.68	-20	—	-20	
HD	347	214.5	1,000.0	16.18	—	+1.0	—	—	—	—	10	16	—	—	+8.19	—	+8.19	+86	—	+86	
ML	240	252.0	115.5	82.4	+1.706	-1.150	-25.97	240	-162	-3,651		109	+2,461	+16.76	-9.42	-25.97	-18.63	-166	1,085	-1,251	
LH	120	612.5	1,000.0	1.96	+2.525	-0.822	-34.16	12	-4	-169		1	+55	+24.80	-6.73	-34.16	-16.09	-59	—	-59	
LN	208	264.0	1,000.0	7.88	-1.755	+0.571	+16.42	24	-8	-227		3	+74	-17.23	+4.68	+16.42	+3.87	+33	—	+33	
NE	369	192.0	1,000.0	19.22	+3.075	—	-28.75	182	—	-1,699		—	—	+30.20	—	-28.75	+1.45	+17	—	+17	
NN'	304	156.0	1,000.0	19.48	-5.050	—	+36.85	497	—	-3,625		—	—	-40.60	—	+36.85	-12.75	-183	—	-183	

$$R_1 \Sigma \lambda^2 = +257,127 R_1$$

$$R_1 \Sigma \lambda \alpha \beta = -46,204 R_1$$

$$\Sigma \lambda \gamma = -2,147,223$$

$$R_2 \Sigma \lambda \alpha \beta = -46,204 R_2$$

$$R_2 \Sigma \lambda \beta^2 = +8,680 R_2$$

$$\Sigma \lambda \beta \gamma = +382,754$$

We can now proceed to determine the loads in the essential bars in terms of the external load system and the redundant

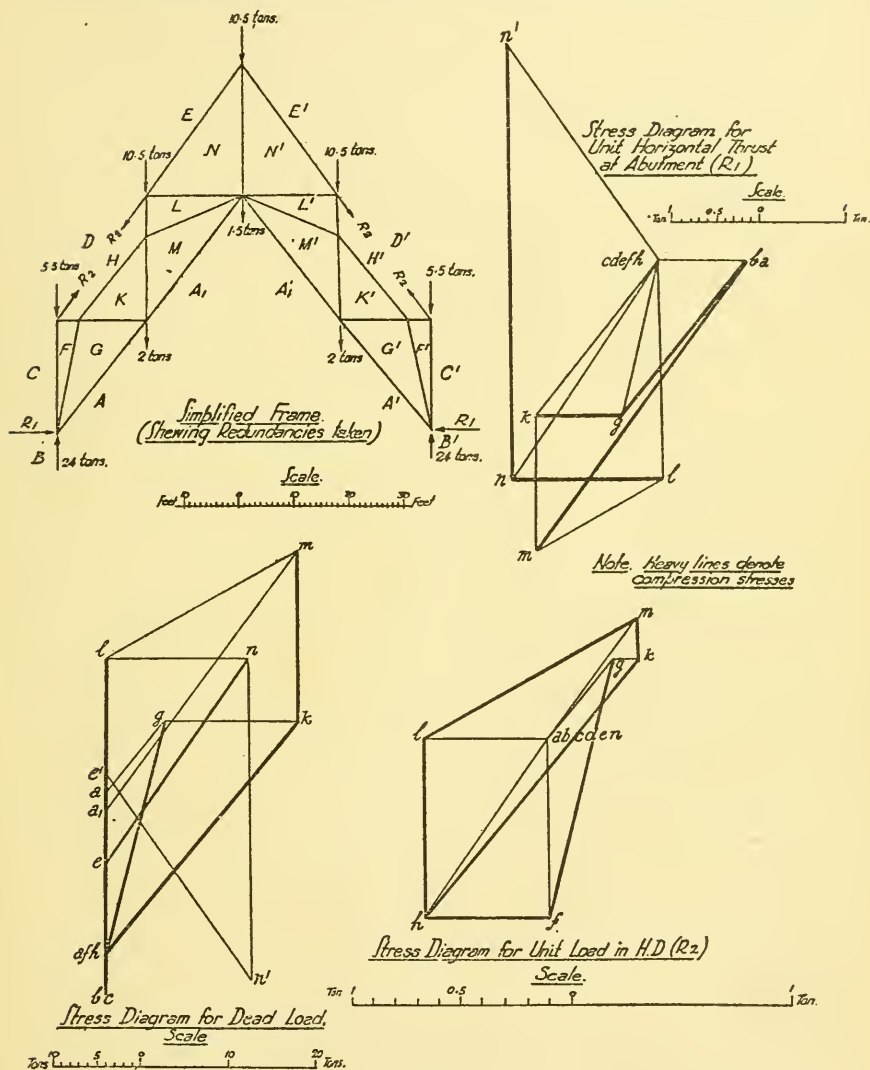


FIG. 63.

forces. This necessitates the drawing of one stress diagram for the external loads, and one each for the redundancies. Since these redundant forces are symmetrical, both in magnitude and

disposition about the centre line of the truss, we only require two extra diagrams—one for the horizontal reaction of the wall  $R_1$  and one for the force  $R_2$  in the member HD. These diagrams are shown in Fig. 63.

The stresses in the various members are given in Table II, cols. 6, 7, and 8, the numbers tabulated being the coefficients of  $R_1$ ,  $R_2$ , and the load due to external forces respectively.

If in any member we denote the load due to the external system by  $\gamma$ , and those due to  $R_1$  and  $R_2$  by  $\alpha R_1$  and  $\beta R_2$ , we have

$$U = \sum \frac{L}{2AE} (\gamma + \alpha R_1 + \beta R_2)^2,$$

$$\frac{\partial U}{\partial R_1} = \sum \frac{L}{AE} (\gamma + \alpha R_1 + \beta R_2) \alpha = 0,$$

and 
$$\frac{\partial U}{\partial R_2} = \sum \frac{L}{AE} (\gamma + \alpha R_1 + \beta R_2) \beta = 0.$$

If we write  $\lambda = \frac{L}{AE}$ , these last two become

$$\sum \lambda \alpha \gamma + R_1 \sum \lambda \alpha^2 + R_2 \sum \lambda \alpha \beta = 0$$

and 
$$\sum \lambda \beta \gamma + R_1 \sum \lambda \alpha \beta + R_2 \sum \lambda \beta^2 = 0.$$

The values of the terms in these expressions are given in cols. 9, 10, and 11 for  $\frac{\partial U}{\partial R_1}$ , and in cols. 12, 13, and 14 for  $\frac{\partial U}{\partial R_2}$ , their sums being given at the bottom of the table.

In arriving at these sums the terms for all members except NN' are written down once only, but are doubled before addition, since the frame is symmetrical.

Substituting the numerical values for the various terms we obtain

$$\begin{aligned} -2,147,223 + 257,127 R_1 - 46,204 R_2 &= 0. \\ 382,754 - 46,204 R_1 + 8,680 R_2 &= 0. \end{aligned}$$

The solution of these gives

$$\begin{aligned} R_1 &= 9.82 \text{ tons} \\ R_2 &= 8.19 \text{ tons.} \end{aligned}$$

The load in any given member due to one of the redundancies is obtained by multiplying that redundant load by its coefficient for the given member, *e.g.* col. 15 is obtained by multiplying  $\alpha$  by  $R_1$  (col. 6 by 9.82) and col. 16 by multiplying  $\beta$  by  $R_2$  (col. 7 by 8.19).

The total load in any member is obtained by adding the dead

load in the simple frame (cols. 8 and 17) to the loads due to the redundancies. This total  $P_0$  is given in col. 18.

For all members direct stresses are obtained by dividing the load in the member by its cross-sectional area. Bending stresses in curved members of small curvature are obtained approximately from the ordinary bending formula :

$$f = \frac{My}{I} = \frac{P_0 R (1 - \cos \phi) y}{A k^2}$$

The total maximum stress in any member is then the sum of the direct and bending stresses in that member.

When the member has a large curvature, the bending stress should be calculated by the application of the formulæ for bent beams, but in this particular example the extra work involved was not considered to be justifiable in view of the nature of the material of which the roof was made.

## EXAMPLE 5

### THE STRESS ANALYSIS OF A BOW GIRDER WITH A CONCENTRATED LOAD

THIS example illustrates the method of analysis described on pp. 41-53, and shows the calculations for the case of a bow girder of semi-elliptical plan form carrying a single concentrated load. This example, with others, was published by the Building Research Station.<sup>1</sup>

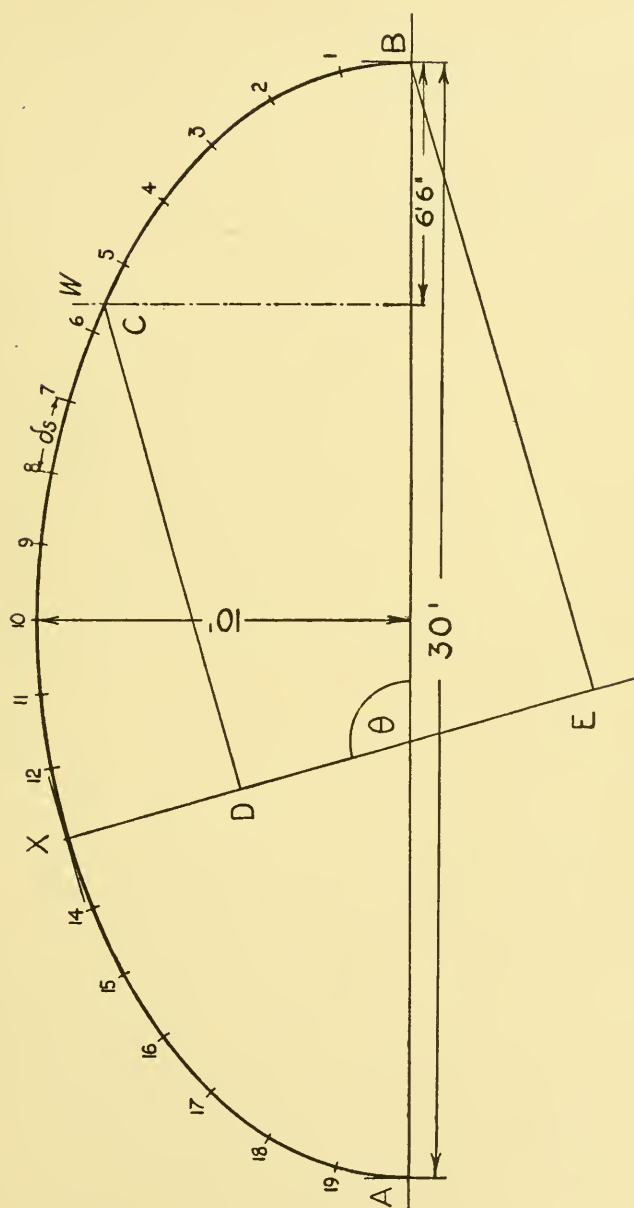
Fig. 64 shows the plan of the centre line of the girder drawn to scale. It is rigidly built in at A and B and carries a load W at the point C. Following the procedure described in Part I, we imagine the girder to be cut at B and the wall reactions replaced by a couple M, a torque T, and a vertical shearing force F, all of which must be evaluated. Divide the perimeter into a number of equal parts: this number should be as large as practicable, since the smaller the intervals between adjacent points the greater will be the degree of accuracy in the final results. At each of these points, numbered 1-19 on the diagram, draw normals to the curve, and from B drop perpendiculars to each of these normals. Thus BE is perpendicular to XE, the normal to the curve at point 13.

From C, the point of application of the load, similar perpendiculars are drawn such as CD. The perpendiculars from C are only required for the normals to points between C and A.

An examination of equations (17), (18), and (19) on pp. 44 and 45 shows that the integrals to be evaluated before M, T, and F can be found involve a number of trigonometrical functions of  $\theta$  and the lengths BE, XE, CD, and XD. A list of the integrals required is given in Table III.

The various terms in these can be obtained from the drawing and the necessary products calculated; a process of graphical

<sup>1</sup> Building Research Technical Paper No. 1. "The Stress Analysis of Bow Girders." A. J. S. Pippard and F. L. Barrow. H.M. Stationery Office, 1926.



integration then enables the terms in the equations to be evaluated.

As an example of the method of obtaining the complete integrals consider the term

$$\int_B^A BE \sin \theta \, ds.$$

The values of the product  $BE \times \sin \theta$  for each point are shown in Table I. The integral required is the area of the curve plotted

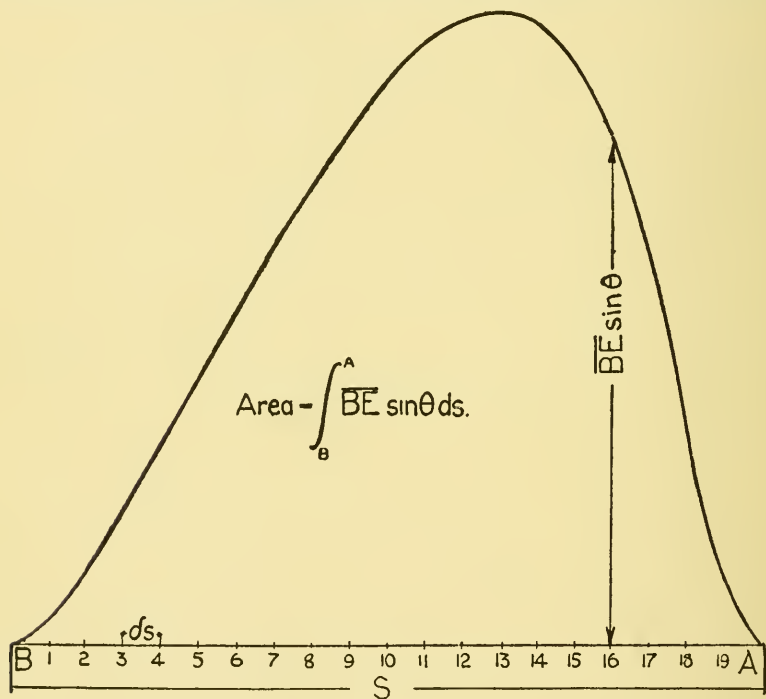


FIG. 65.

on a base equal to the total length of the girder with ordinates equal to  $BE \sin \theta$  at intervals of  $ds$ , erected at the points to which they correspond. Thus the value of  $BE \sin \theta$  for point 16 along the curve is plotted at a distance equal to the length from B to 16 measured round the curve. The area of this diagram is computed either by plotting to scale (Fig. 65) and measuring the area with a planimeter, or, without plotting, by estimation, using Simpson's rule for areas. Table I, col. 6, shows how this calculation may be set out.

In the case of terms such as  $\int_C^A CD \cos \theta ds$  the integration is carried out only for points between C and A. Table II shows the calculation of this integral and Fig. 66 the diagram obtained by plotting.

The calculated values of the integrals, without the coefficients  $\frac{1}{EI}$  and  $\frac{1}{NJ}$ , are given in Table III.

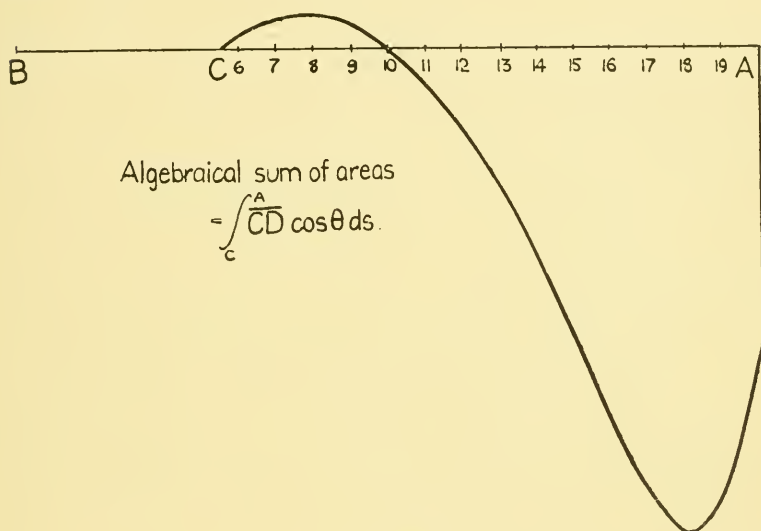


FIG. 66.

**Flexural and Torsional Rigidity.**—The flexural rigidity  $EI = \frac{1}{\alpha}$  and the torsional rigidity  $NJ = \frac{1}{\beta}$  must be considered before the equations involving  $M_0$ ,  $T_0$ , and  $F_0$  can be solved.

As the actual values of  $EI$  and  $NJ$  are not required, but their ratio only, put  $\frac{\beta}{\alpha} = \gamma$ .

Then, dividing equations (17), (18), and (19) by  $\alpha$ , and substituting the numerical values of Table III, the following equations in  $M_0$ ,  $T_0$  and  $F_0$  are obtained :

$$M_0(14.188 + 25.63\gamma) + T_0(0) - F_0(-67.856 + 339.125\gamma) + W(-139.3 + 118.4\gamma) = 0,$$

$$M_0(0) + T_0(25.63 + 14.188\gamma) - F_0(374.8 + 215.33\gamma) + W(243.54 + 143.9\gamma) = 0,$$

$$-M_0(-67.856+339.125\gamma)-T_0(374.8+215.33\gamma) \\ +F_0(6122+8951.9\gamma)-W(4540.3+4726.7\gamma)=0.$$

Gibson and Ritchie, in the work previously referred to on p. 41, take values of  $\gamma$  over the range from 1 to 100. For the purposes of the present example we shall arbitrarily assume  $\gamma=10$ . On substituting this value and solving the three equations we obtain

$$M_0=6.126W, \\ T_0=2.315W, \\ F_0=0.815W.$$

The bending moment, torque and shear force at any point X on the girder may now be found.

Thus between B and C

$$M_x=W(2.315 \sin \theta+6.126 \cos \theta-0.815BE), \\ T_x=W(2.315 \cos \theta-6.126 \sin \theta+0.815XE), \\ F_x=0.815W,$$

and between C and A

$$M_x=W(2.315 \sin \theta+6.126 \cos \theta-0.815BE+CD), \\ T_x=W(2.315 \cos \theta-6.126 \sin \theta+0.815XE-XD), \\ F_x=W(0.815-1)=-0.185W.$$

In these equations the appropriate values of  $\theta$ , BE, CD, XE, and XD, already measured for any point on the girder, may be inserted.

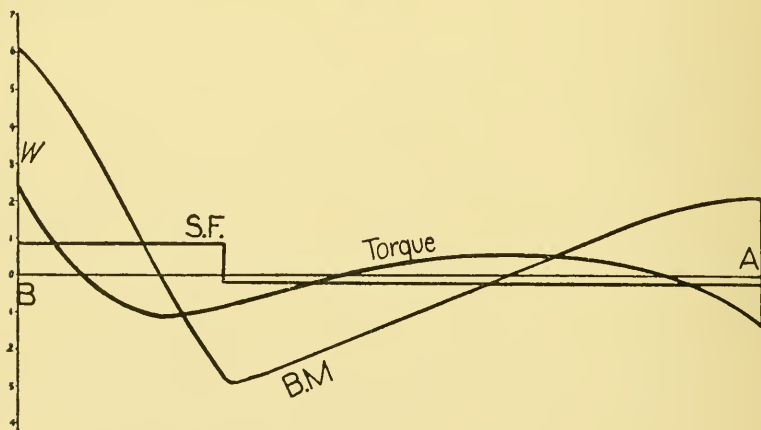


FIG. 67.

Fig. 67 shows the curves of bending moment, torque and shear force throughout the girder, plotted from these expressions.

TABLE I.—EVALUATION OF  $\int_B^A BE \sin \theta \, ds$ .

1	2	3	4	5	6	
Point.	BE	$\theta$	$\sin \theta$	BE $\sin \theta$	Area by Simpson's rule.	
					Multiplying factor.	—
B	0	0	0	0	1	0
1	1.95	15.5	0.2672	0.522	4	2.088
2	3.78	30.0	0.5	1.89	2	3.78
3	5.44	43.8	0.6922	3.74	4	14.96
4	7.00	53.0	0.7986	5.59	2	11.2
5	8.5	60.7	0.872	7.42	4	14.84
6	9.92	67.5	0.9239	9.17	2	18.34
7	11.29	73.8	0.9603	10.84	4	43.36
8	12.6	79.6	0.9835	12.4	2	24.8
9	13.84	84.8	0.9959	13.8	4	55.2
10	15.00	90.0	1.0	15.0	2	30.0
11	16.05	95.2	0.9959	16.0	4	64.0
12	16.92	100.4	0.9835	16.65	2	33.3
13	17.55	106.2	0.9603	16.87	4	67.48
14	17.85	112.5	0.9239	16.5	2	33.0
15	17.66	129.3	0.872	15.4	4	61.6
16	16.95	127.0	0.7986	13.53	2	27.06
17	15.13	136.2	0.6922	10.48	4	41.92
18	11.31	150.0	0.5	5.655	2	11.31
19	6.27	164.5	0.2672	1.68	4	6.72
20	0	180.0	0	0	1	0
						564.96

$$ds = 1.99 \text{ ft.} \quad \frac{ds}{3} = 0.6633.$$

Therefore  $\int_B^A BE \sin \theta \, ds = 0.6633 \times 564.96 = 374.8.$

TABLE II.—EVALUATION OF  $\int_C^A CD \cos \theta ds$ .

1	2	3	4	5	6	
Point.	CD	$\theta$	$\cos \theta$	CD $\cos \theta$	Area by Simpson's rule.	
					Multiplying factor.	—
C <sup>1</sup>	0			0		
6	0.82	67.5	0.3827	0.315	1	0.315
7	2.79	73.8	0.2790	0.778	4	3.112
8	4.75	79.6	0.1805	0.86	2	1.72
9	6.63	84.8	0.0907	0.602	4	2.4
10	8.54	90.0	0	0	1	0
						+ 7.55
10	8.54	90.0	0	0	1	0
11	10.38	95.2	-0.0907	-0.942	4	- 3.76
12	12.07	100.4	-0.1805	-2.18	2	- 4.36
13	13.61	106.2	-0.2790	-3.8	4	- 15.2
14	14.98	112.5	-0.3827	-5.74	2	- 11.5
15	16.06	119.3	-0.4894	-7.9	4	- 31.6
16	16.76	127.0	-0.6018	-10.1	2	- 20.2
17	16.7	136.2	-0.7217	-12.06	4	- 48.24
18	15.39	150.0	-0.87	-13.37	2	- 26.74
19	12.41	164.5	-0.9636	-11.98	4	- 48.0
20	8.23	180.0	-1.0	-8.23	1	- 8.23
						-217.83
						+ 7.55
						-210.28

Now

$$\frac{s}{3} = 0.6633,$$

therefore

$$\int_C^A CD \cos \theta ds = 0.6633 \times (-210.28) \\ = -139.3.$$

<sup>1</sup> The area of the diagram between C and 6 has been neglected, as Simpson's rule has been applied for an odd number of ordinates. The error is seen to be small compared with the total area.

TABLE III.

Integral.	Evaluation.	Integral.	Evaluation.
$\int_B^A \cos^2 \theta \, ds$	14.188	$\int_B^A XE \cos \theta \, ds$	— 215.33
$\int_B^A \sin^2 \theta \, ds$	25.63	$\int_C^A CD \sin \theta \, ds$	243.54
$\int_B^A \sin 2\theta \, ds$	0	$\int_C^A XD \cos \theta \, ds$	— 143.9
$\int_B^A BE \cos \theta \, ds$	— 67.856	$\int_B^A BE^2 \, ds$	6122.0
$\int_B^A XE \sin \theta \, ds$	339.125	$\int_B^A XE^2 \, ds$	8951.9
$\int_O^A CD \cos \theta \, ds$	— 139.3	$\int_O^A CD \cdot BE \, ds$	4540.3
$\int_O^A XD \sin \theta \, ds$	118.4	$\int_O^A XD \cdot XE \, ds$	4726.7
$\int_B^A BE \sin \theta \, ds$	374.8		

## EXAMPLE 6

### THE STRESSES IN A FLYWHEEL

THE determination of the stresses set up in a uniformly rotating flywheel is a problem which is treated in most text-books upon the assumption that the arms are not connected to the rim. This assumption renders the problem a simple one, since, under the action of centrifugal force, the rim is supposed to be subjected only to a pure hoop stress and the arms to a tension varying from a maximum at the nave to zero at the rim; no bending action is allowed for on any part of the rim.

Now the hoop stress in the rim produces a corresponding increase in its length, and so the radius of the wheel is increased; similarly the tensile stress in an arm produces an increase in its length. If the increases in the length of an arm and in the radius were equal, no extra stresses would be set up in the wheel by connecting the rim to the arm. Such, however, is not the case; the increase in wheel radius is greater than the increase in arm length, and there is thus a tendency for them to separate. When there is a good connection between the arm and rim, as in practice there must be, this tendency is counteracted partly by an extra stretch in the arm and partly by a bending inwards of the rim at the point of attachment of the arm.

The complete problem was solved by the late Professor J. G. Longbottom working from the equation to the elastic line of the rim.<sup>1</sup> The following treatment<sup>2</sup> based on the first theorem of Castigliano leads to the same results, but in a somewhat simpler form.

The solution can also be obtained by an application of the principle of least work, as was shown by Professor J. G. Longbottom in the discussion upon the method given in this example.<sup>3</sup>

**General Argument.**—If the arms and rim are free to strain

<sup>1</sup> *Trans. Inst. Engrs. and Shipbuilders in Scotland*, Vol. LXII. 1919.

<sup>2</sup> *Proc. Inst. Mech. Engrs.*, Jan. 1924, "The Stresses in a Uniformly Rotating Flywheel." A. J. S. Pippard.

<sup>3</sup> *Proc. Inst. Mech. Engrs.*, Jan. 1924.

independently when the wheel is rotating, which is the assumption upon which the simple theory is based, the action of centrifugal forces is to cause an unequal alteration in the length of the arms and the radius of the rim; there is assumed to be no bending on the rim and the stresses are easily calculated. If now the arms are forced into contact with the rim by the application of additional radial forces, bending stresses are induced in the rim and extra stresses in the arms. These extra stresses must be superimposed upon those caused by the centrifugal forces, in order to obtain the total resultant stresses in the wheel.

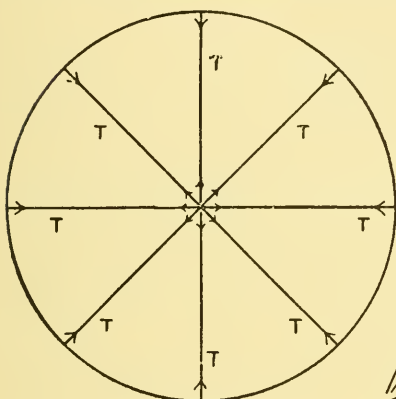


FIG. 68.

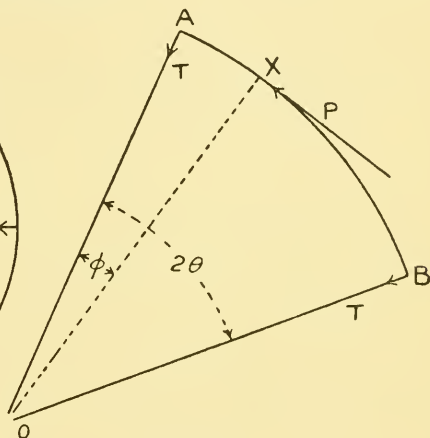


FIG. 69.

It is necessary then to analyse the stresses set up in a circular ring which is connected to a nave by members, which, initially of incorrect length, are forced into position.

Fig. 68 represents a flywheel having  $N$  arms equally spaced and supposed to be rotating at a constant speed. The arms are forced to remain in contact with the rim, and, in order to effect this, an unknown tension  $T$  is required in each arm in addition to the centrifugally induced loads.

If the arms and rim were free to strain independently they would separate by an amount  $\Delta$ . Then if  $U$  be the total strain energy of the wheel

$$\frac{dU}{dT} = N\Delta$$

In Fig. 69 let  $A$  and  $B$  be the points of application of two adjacent loads, and  $O$  the centre of the wheel. Let  $X$  be any point on the rim.

Then, if  $\angle AOB = 2\theta$ ,  
 $\angle AOX = \phi$ ,

the bending moment at X can be shown to be

$$M_x = \frac{TR}{2} \left[ \sin \phi + \cos \phi \cot \theta - \frac{A}{A'\theta} \right] \quad . \quad . \quad (65)$$

where R is the radius to the centroid of the rim, A is the cross-sectional area of the rim and A' is a modified area connected with A to a first approximation<sup>1</sup> by the relation

$$A' - A = \frac{I}{R^2},$$

where I is the moment of inertia of the rim.

Also the tangential component of force at X is

$$P = \frac{T}{2} \operatorname{cosec} \theta \cos (\theta - \phi) \quad . \quad . \quad . \quad (66)$$

There is, in addition, a radial component of force at X, but its effect upon the strain energy can be neglected.

**Calculation of the Strain Energy of the Rim.**—The strain energy of the rim under the action of the system of radial loads comprises three terms :

- (a) the strain energy due to bending,
- (b) the strain energy due to compression by the tangential component of resultant force,
- (c) the small strain energy due to shear which is neglected as usual.

The strain energy due to bending is

$$U_B = \frac{1}{2EI} \int M_x^2 ds,$$

where  $ds$  is an element of the rim perimeter.

Since the loads are evenly spaced, it is only necessary to consider one segment, and, as the value of  $\frac{dU_B}{dT}$  and not  $U_B$  is required, we can, by substituting the value of  $M_x$  from equation (65) and putting  $Rd\phi = ds$ , write the equation

$$\frac{dU_B}{dT} = \frac{TR^3}{4EI} \int_0^{2\theta} \left[ \sin \phi + \cot \theta \cos \phi - \frac{1}{\theta} \right]^2 d\phi,$$

where the factor  $\frac{A}{A'}$ , being practically unity is omitted.

<sup>1</sup> *Strength of Materials*, Morley, chap. xii.

This on integration gives

$$\frac{dU_B}{dT} = \frac{TR^3}{4EI} \left[ \theta \operatorname{cosec}^2 \theta + \cot \theta - \frac{2}{\theta} \right].$$

The strain energy due to direct compression by the tangential component of the resultant force is

$$U_C = \frac{1}{2AE} \int P^2 ds.$$

By substituting the value of  $P$  from equation (66) and integrating, we can write the equation

$$\begin{aligned} \frac{dU_C}{dT} &= \frac{TR \operatorname{cosec}^2 \theta}{8AE} (2\theta + \sin 2\theta) \\ &= \frac{TR}{4AE} [\theta \operatorname{cosec}^2 \theta + \cot \theta]. \end{aligned}$$

Let the length of an arm measured from the nave to the inner edge of the rim be  $l$ .

Then the strain energy in the arm is

$$U_A = \frac{1}{2} \frac{T^2 l}{a_0 E}$$

where  $a_0$  is the cross-sectional area of the arm. If the arm is not uniform, the value of  $U_A$  can be found by a simple graphical integration.

Hence 
$$\frac{dU_A}{dT} = \frac{Tl}{a_0 E},$$

so that, for one segment of the rim and one arm, the complete differential coefficient is the sum of the separate terms found above, that is

$$\frac{dU}{dT} = \frac{T}{E} \left[ (\theta \operatorname{cosec}^2 \theta + \cot \theta) \left( \frac{R^3}{4I} + \frac{R}{4A} \right) - \frac{R^3}{2I\theta} + \frac{l}{a_0} \right]. \quad (67)$$

This expression, equated to  $\Delta$ , the amount of separation during steady rotation of the flywheel assuming there is no connection between the arms and the rim, enables  $T$  to be evaluated.

The amount of this separation must now be found.

**Separation of the Rim from the Arm if Independent Straining Occurs.**—Let the uniform speed of rotation of the wheel be  $\omega$  rads. per sec. Then the hoop stress in the rim is

$$\frac{\rho \omega^2 R^2}{g},$$

where  $\rho$  is the density of the material: the strain is

$$e_1 = \frac{\rho \omega^2 R^2}{gE},$$

and the radius is increased by an amount

$$\frac{\rho \omega^2 R^3}{gE} \quad . \quad . \quad . \quad . \quad . \quad . \quad (68)$$

In Fig. 70, which represents one arm, consider a section distant  $R_x$  from O, the centre of the wheel.

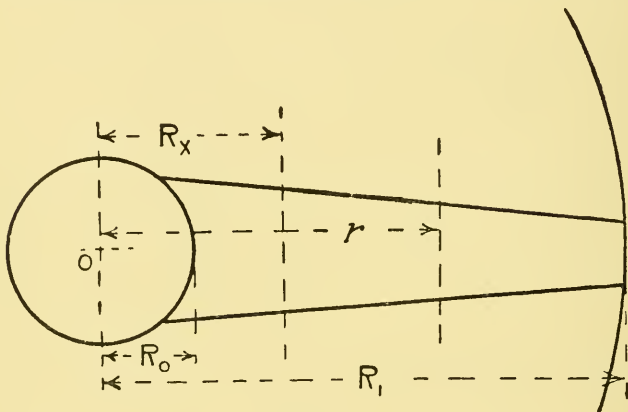


FIG. 70.

At any radius  $r$  let the cross-sectional area of the arm be  $a$ . Then the total force acting across the section  $R_x$  is

$$\frac{\rho \omega^2}{g} \int_{R_x}^{R_1} a r dr,$$

and if  $a_x$  is the area of the arm at  $R_x$ , the strain of the element at  $R_x$  is

$$\frac{\rho \omega^2}{gE a_x} \int_{R_x}^{R_1} a r dr.$$

Hence the increase in length of the element  $dR_x$  is

$$\frac{\rho \omega^2 dR_x}{gE a_x} \int_{R_x}^{R_1} a r dr$$

and the increase in the length of the arm is

$$e = \frac{\rho\omega^2}{gE} \int_{R_0}^{R_1} \frac{dR_x}{a_x} \int_{R_x}^{R_1} ar dr. \quad (69)$$

If the arm is of non-uniform cross-section, this can be evaluated by graphical integration, but if the arm is of uniform cross-sectional area  $a_0$ , equation (69) becomes

$$e = \frac{\rho\omega^2}{6gE} \{2R_1^3 - 3R_0R_1^2 + R_0^3\}. \quad (70)$$

Hence from (68) and (70) it is seen that the arm is too short by the amount

$$\Delta = \frac{\rho\omega^2}{6gE} \{6R^3 - 2R_1^3 + 3R_0R_1^2 - R_0^3\}.$$

Putting  $l = R_1 - R_0$  and equating to  $\frac{dU}{dT}$  from equation (67) we obtain

$$T = \frac{\rho\omega^2 \{6R^3 - 2R_1^3 + 3R_0R_1^2 - R_0^3\}}{6g \left[ (\theta \operatorname{cosec}^2 \theta + \cot \theta) \left( \frac{R^3}{4I} + \frac{R}{4A} \right) - \frac{R^3}{2I\theta} + \left( \frac{R_1 - R_0}{\alpha_0} \right) \right]}. \quad (71)$$

From equation (65) it will be seen that the maximum bending moment on the rim occurs when  $\phi = \theta$ , that is, at the centre of a segment, and its magnitude is

$$M_{\max} = \frac{TR}{2} \left( \operatorname{cosec} \theta - \frac{A}{A'} \frac{1}{\theta} \right). \quad (72)$$

The bending moment at the point of attachment of the arm is

$$M_{\text{arm}} = \frac{TR}{2} \left( \cot \theta - \frac{A}{A'} \frac{1}{\theta} \right). \quad (73)$$

The tangential component of the resultant force also becomes a maximum when  $\phi = \theta$  and is

$$P_{\max} = \frac{T}{2} \operatorname{cosec} \theta. \quad (74)$$

Its value at the point of attachment of the arm is

$$P_{\text{arm}} = \frac{T}{2} \cot \theta. \quad (75)$$

These loads and moments are caused solely by the attachment of the rim and arms. In order to obtain the resultant action at any point in the wheel, the effects due to rotation must be added, these being calculated as if the arms and rim were free to strain independently.

**Actions Due to Rotation, Arms and Rim Independent.**—The forces in the arms due to centrifugal action are tensile, varying from a maximum at the nave to zero at the rim: the magnitude of the load at any point can be calculated as previously explained.

The rim is subjected to a hoop tension of a magnitude  $\frac{\rho}{g} \omega^2 R^2 A$  and also to a uniform bending moment thus:

Consider the rim to be divided into a number of very small segments, each subtending an angle  $2\epsilon$  at the centre of the wheel. Each section exerts a centrifugal force of amount  $-\frac{2\rho}{g} AR^2 \omega^2 \epsilon$  which can be considered as acting through the centre of the segment.

The maximum bending moment in the small segment is then

$$M = -\frac{\rho}{g} AR^3 \omega^2 \left( \epsilon \operatorname{cosec} \epsilon - \frac{A}{A'} \right).$$

In the limit when  $2\epsilon$  becomes infinitely small the conditions are those of a uniformly distributed centrifugal force, and the bending moment becomes

$$M = -\frac{\rho}{g} AR^3 \omega^2 \left( \frac{A' - A}{A'} \right).$$

Now  $A' - A = \frac{I}{R^2}$ , and  $\frac{A}{A'}$  is very nearly unity. Hence

$M = -\frac{\rho}{g} \omega^2 RI$  is the uniform bending moment due to rotation.

Superimposing these centrifugal effects upon the results previously obtained we can write down formulæ for the resultant actions at any point in the wheel.

**Formulæ for Net Resultant Actions.**—The following formulæ then give the total resultant actions at the principal points of the flywheel:

$$\left. \begin{array}{l} \text{Tension} \\ \text{in arm at} \\ \text{junction} \\ \text{with rim} \end{array} \right\} = T = \frac{\rho \omega^2 \{6R^3 - 2R_1^3 + 3R_0 R_1^2 - R_0^3\}}{6g \left[ (\theta \operatorname{cosec}^2 \theta + \cot \theta) \left( \frac{R^3}{4I} + \frac{R}{4A} \right) - \frac{R^3}{2I\theta} + \left( \frac{R_1 - R_0}{a_0} \right) \right]}$$

$$\left. \begin{array}{l} \text{Tension in rim at} \\ \text{arm section} \end{array} \right\} = \left\{ \frac{\rho}{g} \omega^2 R^2 A - \frac{T}{2} \cot \theta \right\}.$$

$$\left. \begin{array}{l} \text{Tension in rim at} \\ \text{mid-section} \end{array} \right\} = \left\{ \frac{\rho}{g} \omega^2 R^2 A - \frac{T}{2} \operatorname{cosec} \theta \right\}.$$

$$\left. \begin{array}{l} \text{Bending moment} \\ \text{at arm section} \end{array} \right\} = \left\{ \frac{TR}{2} \left( \cot \theta - \frac{A}{A'} \frac{1}{\bar{\theta}} \right) - \frac{\rho}{g} \omega^2 RI \right\}.$$

$$\left. \begin{array}{l} \text{Bending moment} \\ \text{at mid-section} \end{array} \right\} = \left\{ \frac{TR}{2} \left( \operatorname{cosec} \theta - \frac{A}{A'} \frac{1}{\bar{\theta}} \right) - \frac{\rho}{g} \omega^2 RI \right\}.$$

In calculating bending stress in the rim, it should be remembered that the formulæ for straight beams are inapplicable. Those for initially-curved bars must be used.



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THE END

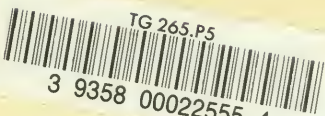






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Pippard, Alfred John Sutton, 1891-  
Strain energy methods of stress  
analysis, by A. J. Sutton Pippard.  
London, New York [etc.] Longman's,  
Green and co. ltd., 1928.  
x, 146 p. pl., diagsr. 24 cm.

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